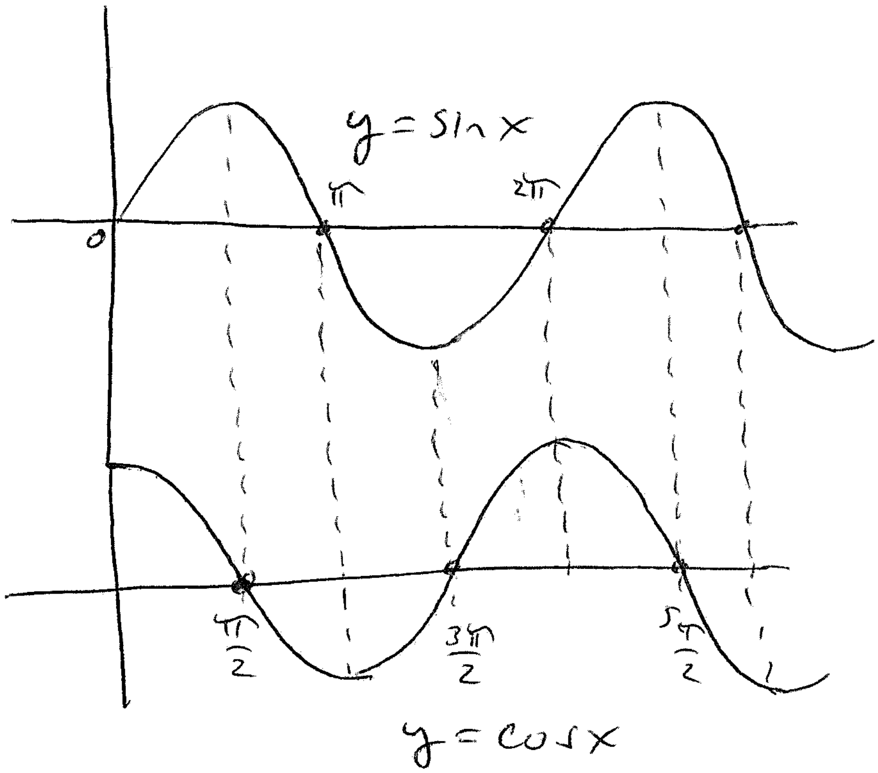
Class 18 : [redacted] Section 4.6 I



Note the relationship between the properties of the graph of $y = \sin x$ with those of $y = \cos x$.

- $y = \cos x$ seems to have a 0 precisely where $y = \sin x$ has a horizontal tangent line.
- The places of $y = \sin x$ where it is rising the fastest or dropping the most correspond to the highest (respectively lowest) parts of $y = \cos x$
- The derivative of $y = \sin x$ looks like it exists everywhere, like $y = \cos x$.

Studying the two graphs, one can make a guess that $\frac{d}{dx}[\sin x] = \cos x$.

The values of the derivative, as a function, document the slopes of the tangent lines of the graph of the function at corresponding points.

In fact, $\frac{d}{dx}[\sin x] = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$

But $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$. Hence

$$\frac{d}{dx}[\sin x] = \lim_{h \to 0} \frac{(\sin(x)\cos(h) + \sin(h)\cos(x)) - \sin x}{h}$$

$$= \lim_{h \to 0} \left[\sin(x)\left(\frac{\cos(h) - 1}{h}\right) + \cos(x)\left(\frac{\sin(h)}{h}\right) \right]$$

Sum ~~Prod and~~ rule for limits

$$= \lim_{h \to 0} \left(\sin(x)\left(\frac{\cos(h) - 1}{h}\right)\right) + \lim_{h \to 0} \left(\cos(x)\left(\frac{\sin(h)}{h}\right)\right)$$

prod rule

$$= \underbrace{\lim_{h \to 0} (\sin x)}_{\sin(x)} \cdot \underbrace{\left(\lim_{h \to 0} \frac{\cos(h) - 1}{h}\right)}_{0} + \underbrace{\lim_{h \to 0} \cos(x)}_{\cos(x)} \cdot \underbrace{\lim_{h \to 0} \left(\frac{\sin(h)}{h}\right)}_{1}$$

$$= \cos(x).$$

One could do the same calculation to show that
$$\frac{d}{dx}[\cos x] = -\sin x$$

Do this!!!

Other examples

- $\frac{d}{dx}[\sin \frac{\pi}{2}] = 0$ (it is a constant).

- $\frac{d}{dx}[\cos x^2] = (-\sin x^2)(2x) = -2x\sin(x^2)$
  by the Chain Rule

- $\frac{d}{dx}[\tan x] = \frac{d}{dx}\left[\frac{\sin x}{\cos x}\right] \underset{\text{Rule}}{\overset{\text{Quotient}}{=}} \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{(\cos x)^2}$
  $$= \frac{1}{(\cos x)^2} = (\sec x)^2 = \sec^2 x.$$

- $\frac{d}{dx}[\cos^2 x] = \frac{d}{dx}[(\cos x)^2] = 2(\cos x)(-\sin x)$

  (see the difference between this one and $\frac{d}{dx}[\cos x^2]$?

Other examples cont'd.

- $\frac{d}{dx}[\sec x] = \frac{d}{dx}\left[\frac{1}{\cos x}\right] = \frac{0(\cos x) - 1(-\sin x)}{(\cos x)^2}$

$$= \frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} = \tan x \sec x.$$

again by the Quotient Rule.

- $\frac{d}{dx}[\cot x] = ?$
- $\frac{d}{dx}[\csc x] = ?$

$\Big\}$ work these out!

How about $\frac{d}{dx}\left[\sin\left(\sqrt{x^3+2x}\right)\right]$?

Let
$$\left.\begin{aligned} h(x) &= x^3 + 2x \\ g(x) &= \sqrt{x} \\ f(x) &= \sin x \end{aligned}\right\} \quad f(g(h(x))) = \sin\sqrt{x^3+2x}$$

Then $\frac{d}{dx}\left[\sin\sqrt{x^3+2x}\right] = \frac{d}{dx}\left[f(g(h(x)))\right]$

$$= f'(g(h(x))) \cdot g'(h(x)) \cdot h(x)$$

$$= \cos\left(\sqrt{x^3+2x}\right) \cdot \frac{1}{2\sqrt{x^3+2x}} \cdot (3x^2+2)$$

Another type of differentiable function:

Exponential functions

Let $f(x) = a^x$, where $a > 0$, $a \neq 1$.

It should be clear that $\frac{d}{dx}[a^x] \neq x a^{x-1}$

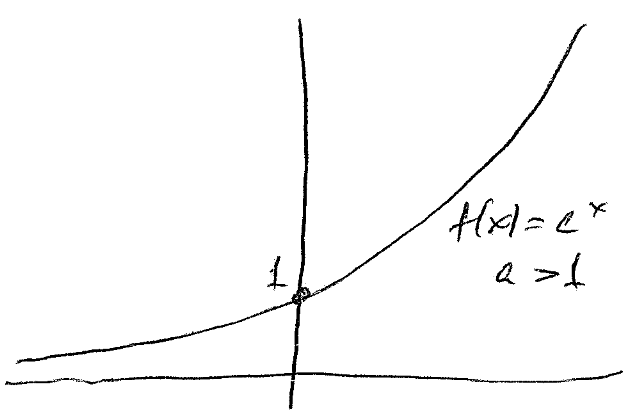
since if $x = 1$, then

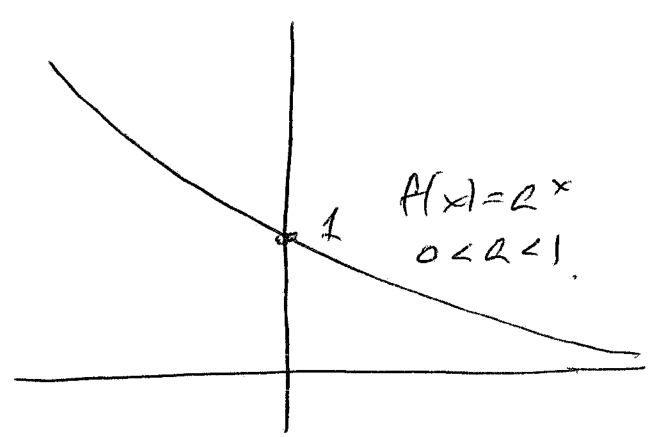
$$f'(1) = \frac{d}{dx}[a^1] = 0 \text{ since } a \text{ is a constant,}$$

but

$$x a^{x-1}\Big|_{x=1} = 1 a^{1-1} = 1 a^0 = 1.$$

So what is $\frac{d}{dx}[a^x]$? It looks from the graph that this should exist for all x:





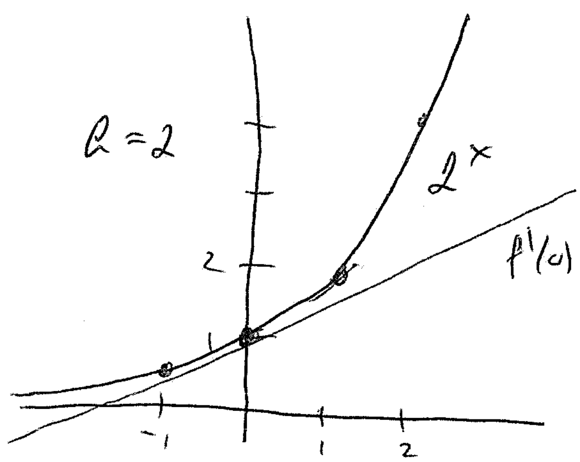
By the definition:

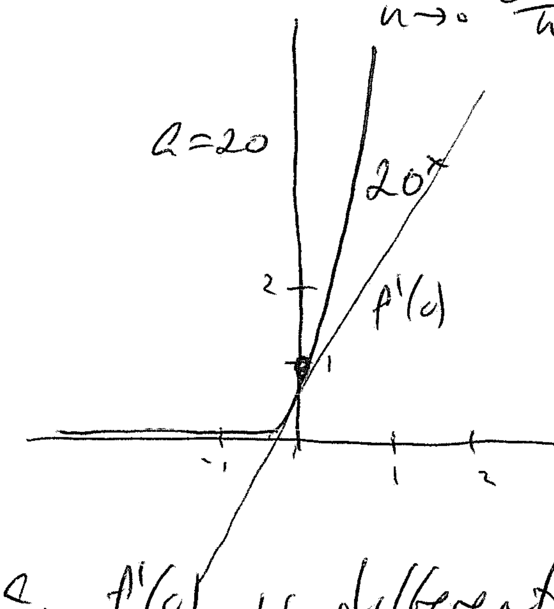
$$f'(x) = \frac{d}{dx}[a^x] = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h \to 0} \frac{a^x a^h - a^x}{h}$$

$$= \lim_{h \to 0} a^x \left(\frac{a^h - 1}{h} \right) \overset{\text{const mult}}{\underset{\text{Rule}}{=}} a^x \left(\lim_{h \to 0} \frac{a^h - 1}{h} \right)$$

This already says that the derivative of an exponential function is the same exponential function times a constant (as long as $\left(\lim_{h \to 0} \frac{a^h - 1}{h} \right)$ exists, that is).

But this quantity is simply $f'(0) = \lim_{h \to 0} \frac{a^{0+h} - a^0}{h}$
$= \lim_{h \to 0} \frac{a^h - 1}{h}$.



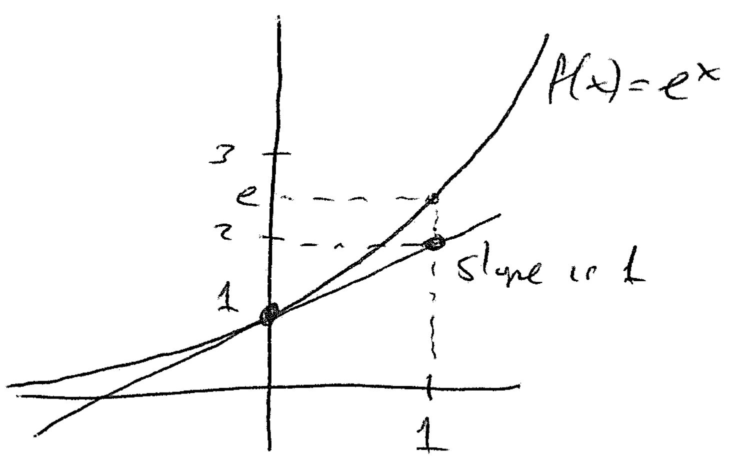


For different values of a, $f'(0)$ is different.

Note: There is one (of the various) definitions of the number $e \approx 2.71828\ldots$, as the precise number that satisfies

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

Here e is the base of the exponential function whose derivative at 0 is precisely 1.



This immediately implies the following:

Let $f(x) = e^x$. Then

$$f'(x) = \frac{d}{dx}[e^x] = e^x \left(\lim_{h \to 0} \frac{e^h - 1}{h} \right) = e^x f'(0) = e^x \cdot 1 = e^x$$

$$\frac{d}{dx}[e^x] = e^x$$

Can you think of other functions where the derivative equals the function itself?