Class 19: Section 4.7 I

Last class we showed: For $f(x)=a^x$

$$\frac{d}{dx}[f(x)] = a^x f'(0), \text{ where } f'(0)=\lim_{h\to 0}\frac{a^h-1}{h}.$$

and that $e \doteq 2.71828$ was defined as the value for a so that $\lim_{h\to 0}\frac{a^h-1}{0} = 1$.

$$\Longrightarrow \quad \frac{d}{dx}[e^x] = e^x \lim_{h\to 0}\frac{e^h-1}{h} = e^x(1) = e^x$$

Notes ① How many other functions can you think of where the derivative function equals the function itself

② $$\frac{d}{dx}\left[e^{f(x)}\right] \underset{\text{Rule}}{\overset{\text{chain}}{=\!=}} e^{f(x)} \cdot f'(x)$$

ex: $$\frac{d}{dx}\left[e^{\sin x}\right] = e^{\sin x}\cdot \cos x$$

$$\frac{d}{dx}\left[e^{2x^2+3x}\right] = e^{2x^2+3x}\cdot(4x+3)$$

Notes cont'd

③ $\frac{d}{dx}[a^x] = ?$ Rewrite $a^x = e^{\ln a^x}$

(Remember, $\ln a^x$ only works when $a^x > 0$. But a^x is always > 0 !!!)

$a^x = e^{\ln a^x} = e^{x \ln a}$, where $\ln a$ is a constant.

Let $f(x) = x \ln a$. Then $f'(x) = \ln a$.

And $\frac{d}{dx}[a^x] = \frac{d}{dx}[e^{x \ln a}] = \frac{d}{dx}[e^{f(x)}]$

$= e^{f(x)} \cdot f'(x) = e^{x \ln a} \cdot \ln a$

$$\boxed{\frac{d}{dx}[a^x] = a^x \ln a}$$

④ Recall that $\frac{d}{dx}[a^x] = a^x \lim_{h \to 0} \frac{a^h - 1}{h}$. Then

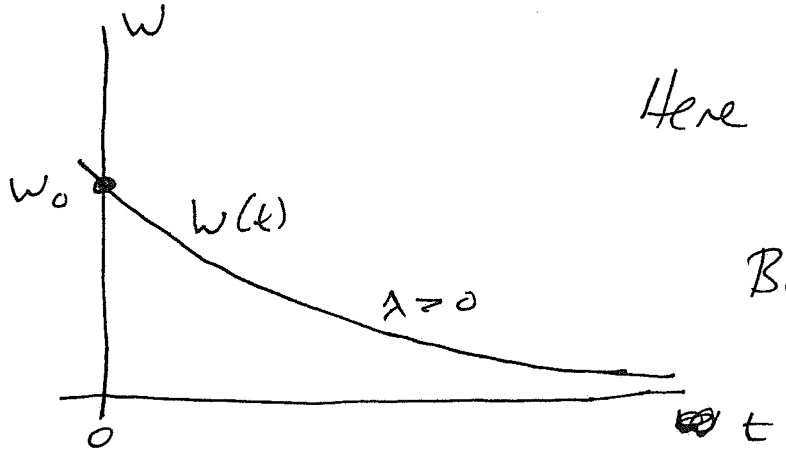
$$\lim_{h \to 0} \frac{a^h - 1}{h} = \ln a.$$

Hence for $a = e \doteq 2.71828 \cdots$,

$$\lim_{h \to 0} \frac{e^h - 1}{h} = \ln e = 1.$$

Here is an application

Let $W(t) = W_0 e^{-\lambda t}$, an exponential function representing radioactive decay of some substance (Note the exponent is a linear fnc of t)



Here $\frac{dW}{dt} = W_0 (-\lambda) e^{-\lambda t}$

But $W_0 e^{-\lambda t} = W(t)$. Hence

$$\frac{dW}{dt} = (-\lambda) W(t)$$

- This means for this type of function, the rate of decay is proportional to the amount of material present.
- The derivative of an exponential function is always a constant times the original function

- New question: Find a function whose derivative satisfies the equation

$$\frac{dy}{dt} = ry$$

Since this equation states: the derivative of some unknown function $y(t)$ is a constant times the original function.

The answer is: $y(t) = C_0 e^{rt}$. Does this work?

$$y'(t) = \frac{dy}{dt} = \frac{d}{dt}\left[C_0 e^{rt}\right] = C_0 \frac{d}{dt}\left[e^{rt}\right]$$
$$= C_0 e^{rt}(r)$$
$$= r(C_0 e^{rt}) = ry. \blacksquare$$

Back to inverse functions

Recall, if a function $f(x)$ has an inverse, then there exists a function $f^{-1}(x)$, where

domain of $f^{-1}(x)$ = range of $f(x)$

range of $f^{-1}(x)$ = domain of $f(x)$

Also, if $f^{-1}(x)$ exists, then

$$(f \circ f^{-1})(x) = f(f^{-1}(x)) = x, \text{ and}$$
$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x.$$

ex. Let $f(x) = x^2$ on the domain $[0, \infty)$.

Its inverse is $f^{-1}(x) = \sqrt{x}$ on $[0, \infty)$.

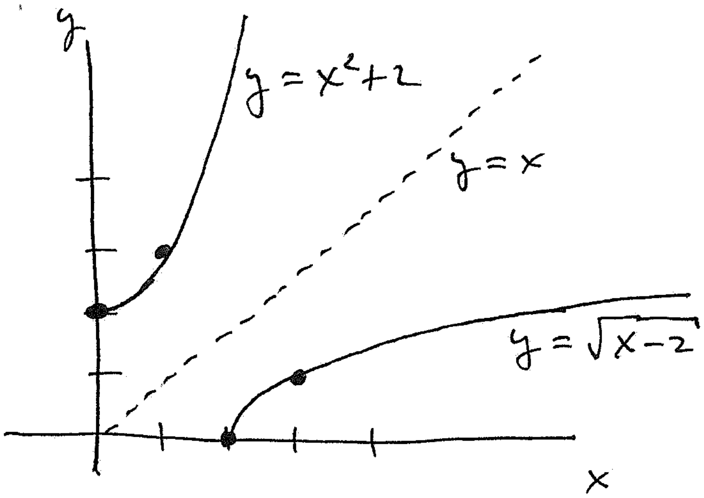
Here $(f \circ f^{-1})(x) = f(\sqrt{x}) = (\sqrt{x})^2 = x$ on $[0, \infty)$

and $(f^{-1} \circ f)(x) = f^{-1}(x^2) = \sqrt{x^2} = x$ on $[0, \infty)$.

ex. Also $f(x) = e^x$, $f^{-1}(x) = \ln x$. Domain? Range?

Sometimes we can find $f^{-1}(x)$ by solving for x as a function of y and then switching the variables.

ex. For $y = x^2 + 2$ on $[0, \infty)$, solve for ~~y~~ x as a function of y: $x = \pm\sqrt{y-2}$. This is not a function, but restricted to ~~range~~ $x \in [0, \infty)$, only the positive part of the parabola is represented in the original function. Hence $x = \sqrt{y-2}$. To graph, switch x, y. $\boxed{y = \sqrt{x-2}}$.



- Notice the symmetry with respect to the $y=x$ line of $f(x)$ and $f^{-1}(x)$, when it exists.

Q: How are the derivatives of $f(x)$ and $f^{-1}(x)$ related?

Note: This is good to know in the case where the derivative of $f^{-1}(x)$ is needed but one cannot calculate by solving for x as a fnc of y.

Suppose $f(x)$ has an inverse. Find $\frac{d}{dx}[f^{-1}(x)]$.

Use the Chain Rule: Since $(f \circ f^{-1})(x) = x$, then

$$\frac{d}{dx}[(f \circ f^{-1})(x)] = \frac{d}{dx}[x] = 1.$$

$$\parallel$$

$$\frac{d}{dx}[f(f^{-1}(x))] = f'(f^{-1}(x)) \cdot \frac{d}{dx}[f^{-1}(x)] = 1.$$

Hence $$\boxed{\frac{d}{dx}[f^{-1}(x)] = \frac{1}{f'(f^{-1}(x))}}$$ What does this mean?

Notes ① This is a form of implicit differentiation.
One can use this formula without actually knowing $f^{-1}(x)$.

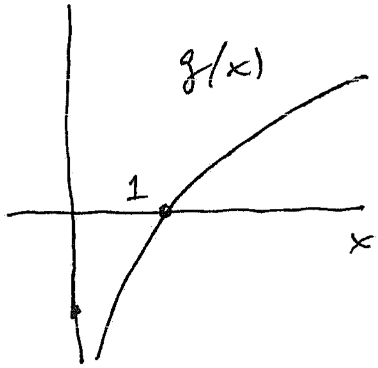
② For $x = f^{-1}(y)$, the inverse to $y = f(x)$, then in Leibniz notation $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$

③ Suppose we did not know $\frac{d}{dx}[\sqrt{x}]$.
But we knew $y = f^{-1}(x) = \sqrt{x}$ is the inverse to $y = f(x) = x^2$. Then

$$\frac{d}{dx}[\sqrt{x}] = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(\sqrt{x})} = \frac{1}{2\sqrt{x}}$$

since $f'(x) = 2x$.

④ So what is the derivative of $g(x) = \ln x$?



Here $g(x)$ is the inverse to $y = e^x$. Hence $g^{-1}(x) = e^x$

Notes cont'd.

(4) cont'd So use the previous equation:

$$x = (g^{-1} \circ g)(x) = (g \circ g^{-1})(x)$$
$$= e^{\ln x} \qquad = \ln e^{x}$$

Since $x = e^{\ln x}$, $\frac{d}{dx}[x] = \frac{d}{dx}[e^{\ln x}]$

$$1 = e^{\ln x} \cdot \frac{d}{dx}[\ln x]$$
$$= x \cdot \frac{d}{dx}[\ln x].$$

Solve for the derivative:

$$\boxed{\frac{1}{x} = \frac{d}{dx}[\ln x]}$$

(5) Notice that now we can do the same thing for other base logs

$\frac{d}{dx}[\log_a x]$: Here $x = a^{\log_a x}$, so

$$\frac{d}{dx}[x] = \frac{d}{dx}[a^{\log_a x}] =$$
$$\overset{\shortparallel}{1} \qquad = \left[(a^{\log_a x})(\ln a)\right]\frac{d}{dx}[\log_a x]$$
$$= x \ln a \, \frac{d}{dx}[\log_a x].$$

Hence $\boxed{\frac{d}{dx}[\log_a x] = \frac{1}{x \ln a}}$

Notes cont'd

⑥ How about $\frac{d}{dx}[\ln f(x)]$, when $f(x)$ is a differentiable function?

$$\frac{d}{dx}[\ln f(x)] = \underbrace{\frac{1}{f(x)}}_{\substack{\text{derivative of ln} \\ \text{fnc evaluated} \\ \text{at } f(x)}} \cdot f'(x) = \frac{f'(x)}{f(x)}$$

ex. $\frac{d}{dx}[\ln(x^2+3x)] = \frac{1}{x^2+3x} \cdot (2x+3) = \frac{2x+3}{x^2+3x}$

ex. $\frac{d}{dx}[\ln \cos x] = \frac{1}{\cos x}(-\sin x) = -\tan x$

ex. $\frac{d}{dx}[\log_2 \sqrt{x^2+2x}] = \frac{1}{\sqrt{x^2+2x}\,\ln 2} \cdot \frac{1}{2\sqrt{x^2+2x}} \cdot (2x+2)$

$= \frac{x+1}{(x^2+2x)\ln 2}$ Can you see this?