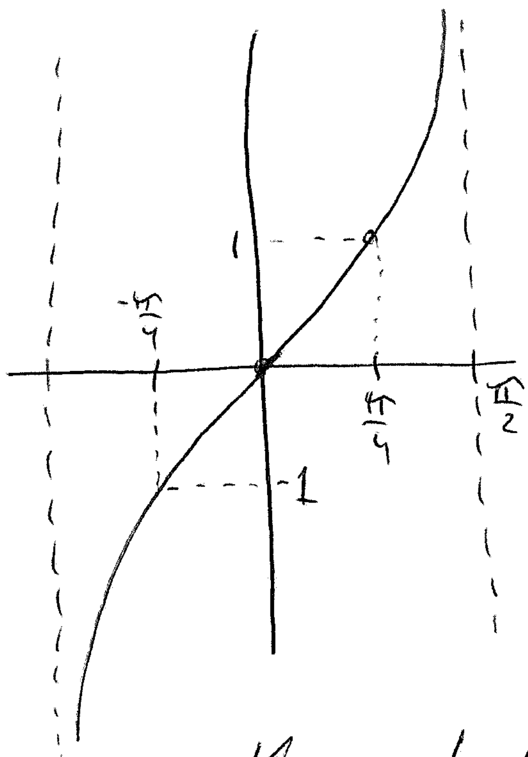
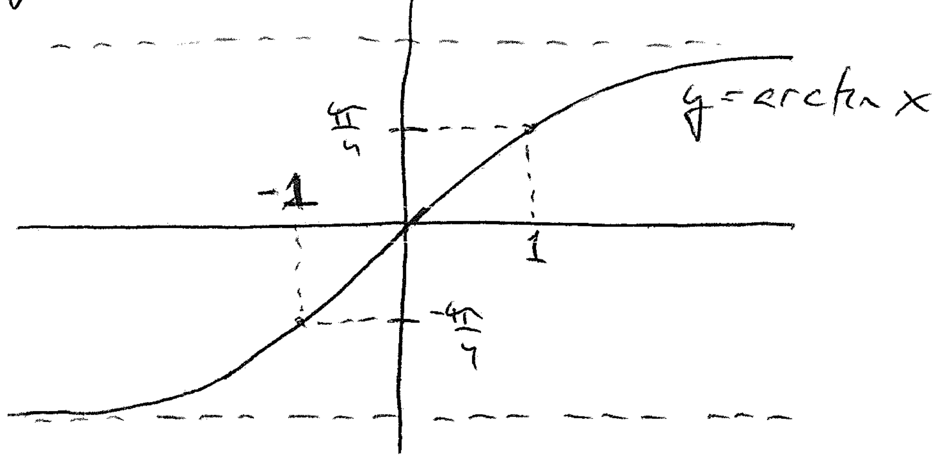
Class 20: [redacted] Section 4.7



Still with inverse functions, it is easy to see visually if a function $f(x)$ on some domain has an inverse: if it satisfies the horizontal rule (so that its inverse will satisfy the vertical line rule as a function).

The function $y = \tan x$ above (on $(-\frac{\pi}{2}, \frac{\pi}{2})$) has an inverse, denoted either $y = \tan^{-1} x$ or $y = \arctan x$ on domain $(-\infty, \infty)$



Check that $y = \arctan x$ is just the reflection of $y = \tan x$ across the $y = x$ line.

Notice that $\frac{d}{dx}[\tan x] = \sec^2 x$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$.
Hence $y = \arctan x$ should also have derivatives. So what is $\frac{d}{dx}[\tan^{-1} x]$?

This takes a bit of work....

Note that $\underline{y = \arctan x}$ is the same as $\underline{x = \tan y}$

so that the composition $\boxed{x = \tan y = \tan(\arctan x)}$ makes sense.

Differentiate what is in the box to help construct $\frac{d}{dx}[\arctan x]$....

$$\frac{d}{dx}[x] = \frac{d}{dx}\left[\tan(\arctan x)\right]$$

$$1 = \sec^2(\underbrace{\arctan x}_{y}) \cdot \frac{d}{dx}[\arctan x]$$

So that $$\frac{1}{\sec^2 y} = \frac{d}{dx}[\tan^{-1} x].$$

This is okay, but to really see what the derivative of $\arctan x$ is, we need to write the left side in terms of x:

Here $$\frac{d}{dx}[\tan^{-1} x] = \frac{d}{dx}[\arctan x]$$
$$= \frac{1}{\sec^2 y} = \frac{1}{1 + \underbrace{\tan^2 y}_{x^2}}$$ by an identity.

and since $x = \tan y$, we get
$$\frac{d}{dx}[\tan^{-1} y] = \frac{d}{dx}[\arctan x] = \frac{1}{1+x^2}$$

<u>HW</u> Do the same calculation for $y = \sin^{-1} x$ and $y = \cos^{-1} x$.

Hint: For $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\cos y \geq 0$, and
$$\cos y = \sqrt{1 - \sin^2 y}.$$

Yet one more use of the Chain Rule

Q: How does one study a function of the form

$$y = (f(x))^x \quad \text{[neither a poly nor an exponential]}$$

A: <u>Method 1</u>: If we knew $(f(x))^x$ were always positive, then

$$(f(x))^x = e^{\ln (f(x))^x} = e^{x \ln f(x)}.$$

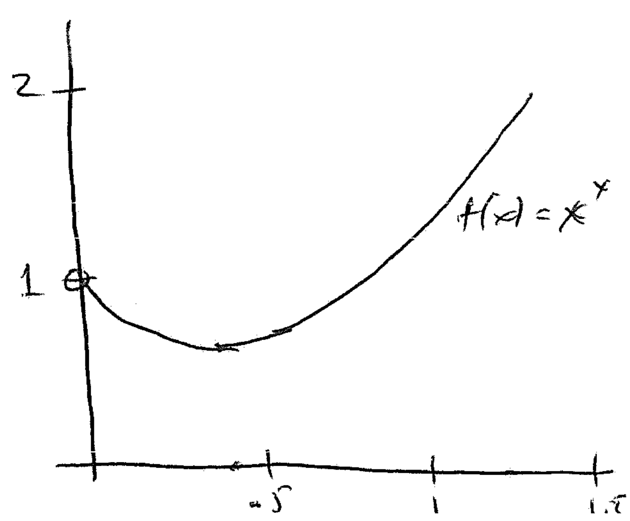
<u>ex</u> Let $y = x^x$ ~~[illegible]~~ on the domain $(0,\infty)$

$$\frac{dy}{dx} = \frac{d}{dx}[x^x] = \frac{d}{dx}\left[e^{\ln x^x}\right] = \frac{d}{dx}\left[e^{x\ln x}\right].$$

$$= e^{x\ln x}\cdot \frac{d}{dx}[x\ln x]$$

$$= e^{x\ln x}\cdot \overbrace{\left(1\cdot \ln x + x\cdot\left(\tfrac{1}{x}\right)\right)}^{\text{Prod rule}}$$

$$= x^x(\ln x + 1).$$



Method 2 Create a new function and differentiate the new one, leading to a solution of the original problem:

Instead of $y = x^x$, where $x^x > 0$, create

$\ln y = \ln x^x$ and differentiate this:

$$\underbrace{\frac{d}{dx}[\ln y]} = \frac{d}{dx}[\ln x^x] = \frac{d}{dx}[x \ln x].$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = 1 \cdot \ln x + x \cdot \left(\frac{1}{x}\right) = (\ln x + 1)$$

Now solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = y(\ln x + 1) = x^x(\ln x + 1)$$

Same result. This method is called

<u>logarithmic differentiation</u>.

This new method is good for

- functions with the variable in both the base and the exponent.
- Rational-looking function with a lot of factors.

(logarithms take products to sums)

$$\log(ab) = \log a + \log b$$
$$\log\left(\frac{a}{b}\right) = \log a - \log b.$$

ex pg 191 Example 12.

Find $\frac{d}{dx}\left[\frac{e^x x^{3/2}\sqrt{1+x}}{(x^2+3)^4(3x-2)^3}\right]$.

Strategy: Use logarithmic differentiation to simplify product and ratio.

For $y = \frac{e^x x^{3/2}\sqrt{1+x}}{(x^2+3)^4(3x-2)^3}$,

$$\ln y = \ln\left(\frac{e^x x^{3/2}\sqrt{1+x}}{(x^2+3)^4(3x-2)^3}\right) =$$
$$= \ln e^x + \ln x^{3/2} + \ln\sqrt{1+x} - \ln(x^2+3)^4 - \ln(3x-2)^3$$

Example (cont'd.).

Solution

$$\ln y = x + \frac{3}{2}\ln x + \frac{1}{2}\ln(1+x) - 4\ln(x^2+3) - 3\ln(3x-2)$$

and $$\frac{d}{dx}[\ln y] = \frac{1}{y}\cdot\frac{dy}{dx} = 1 + \frac{3}{2x} + \frac{1}{2(1+x)} - \frac{4(2x)}{x^2+3} - \frac{3(3\cancel{x})}{3x-2}$$

and $$\frac{dy}{dx} = y\left(1 + \frac{3}{2x} + \frac{1}{2(1+x)} - \frac{8x}{x^2+3} - \frac{9}{3x-2}\right)$$

$$= \frac{e^x x^{3/2}\sqrt{1+x}}{(x^2+3)^4(3x-2)^3}\left(1 + \frac{3}{2x} + \frac{1}{2(1+x)} - \frac{8x}{x^2+3} - \frac{9}{3x-2}\right)$$

<u>From the exam:</u>

$$f(x) = \frac{\sqrt{x^2+8x}}{4x}\text{, just for } x>0\text{. part. Calculate } f'(x).$$

Let $y = \frac{\sqrt{x^2+8x}}{4x}$. Then

$$\ln y = \frac{1}{2}\ln(x^2+8x) - \ln(4x)$$

$$\frac{d}{d\cancel{x}}[\ln y] = \frac{1}{y}\frac{dy}{dx} = \frac{2x+8}{2(x^2+8x)} - \frac{4}{4x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{x^2+8x}}{4x}\left(\frac{x+4}{x^2+8x} - \frac{1}{x}\right)$$