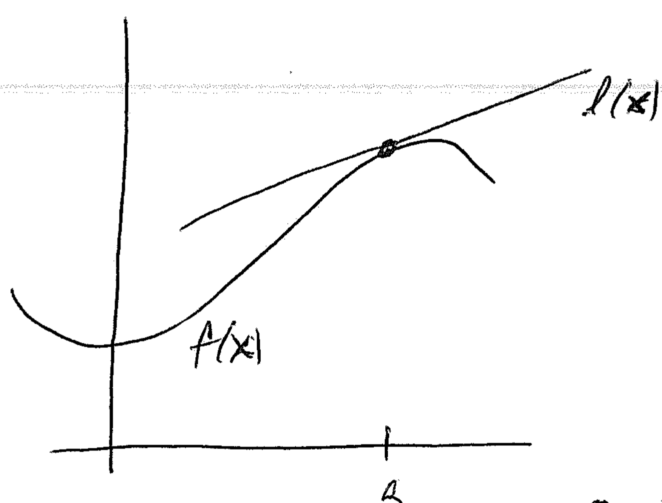
Class 21: Section 5.1

Q: Suppose you needed to know $\sqrt{110}$ but did not have a calculator. What is your most accurate guess? How can you be accurate?

The tangent line to a pt $(a, f(a))$ on the graph of a function $f(x)$ is <u>called</u> the <u>best</u> <u>linear</u> <u>approximation</u> to the curve at a.



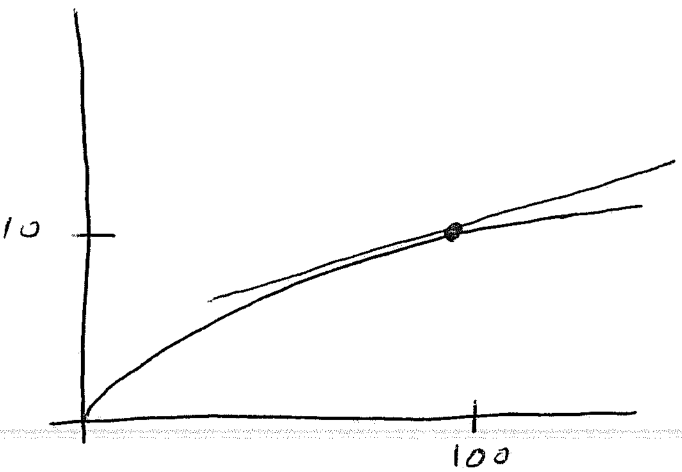
- The equation of the line $l(x) = f(a) + f'(a)(x-a)$
- $l(x)$ has the same value at $x=a$: $l(a) = f(a)$
- $l(x)$ has the same derivative at $x=a$: $l'(a) = f'(a)$.

Hence, for values of x "near" a, $l(x)$ and $f(x)$ are very close.

ex. Let $f(x) = \sqrt{x}$. Here $f(110) = \sqrt{110}$ is the exact value you are looking for. But how to calculate it?

There is a value of $\sqrt{x}$ nearby that is easy to calculate: $f(100) = \sqrt{100} = 10$.

The ~~[illegible]~~ tangent line to $f(x)$ at $x = 100$ is



$$l(x) = f(100) + f'(100)(x - 100)$$

$$= 10 + \frac{1}{2\sqrt{100}}(x - 100)$$

$$= 10 + \frac{1}{20}(x-100) = 10 + \frac{1}{20}x - 5 = \frac{1}{20}x + 5$$

Since the tangent line to $f(x)$ at $x = 100$ is close to $f(x)$ at $x = 110$, $l(110) \approx f(110)$

$$10.5 = \frac{1}{20}(110) + 5 \approx \sqrt{110}$$

Actual value is closer to $f(110) = \sqrt{110} \triangleq 10.4881\ldots$

HW Do the same for the known value $x = 121$.

Def For $y = f(x)$ differentiable at $x = a$, the line

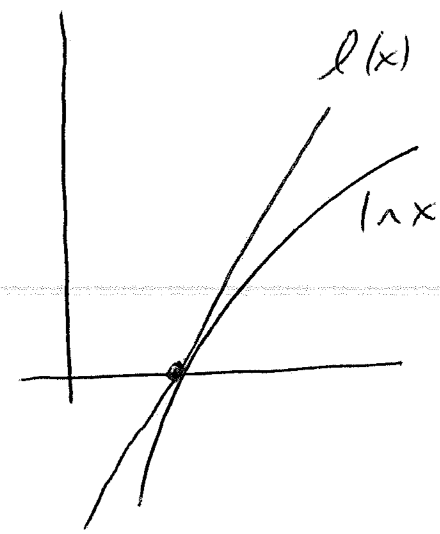
$$\ell(x) = f(a) + f'(a)(x-a)$$

is called the tangent line approximation or the local linearization of $f(x)$ at $x = a$.

ex. Estimate $\ln(1.1)$ using the tangent line approximation of $f(x) = \ln x$ at $x = 1$.

Solution: Here, $f(1) = 0$ and

$$f'(1) = \left.\frac{1}{x}\right|_{x=1} = 1.$$



Hence

$$\begin{aligned} \ell(x) &= f(1) + f'(1)(x-1) \\ &= 0 + 1(x-1) \\ &= x-1 \end{aligned}$$

And $\cancel{\ell(x)} = \ell(1.1) = 1.1 - 1 = .1 \approx f(1.1) = \ln 1.1$

The actual value is closer to $.09531$.

Note: There are ways to measure the accuracy of the local linearization. We will not do this here.

Def Let f be defined on a domain D and let $c \in D$. Then f has a <u>global maximum</u> at $x=c$, if $f(x) \leq f(c)$ for all $x \in D$. $f(x)$ has a <u>global minimum</u> at $x=c$ if $f(x) \geq f(c)$ for all $x \in D$.

Note: ① A maximum or a minimum is also called an <u>extremum</u>.
② The global extremum $f(c)$ is a <u>function value</u>.
③ The domain D will be very important here.

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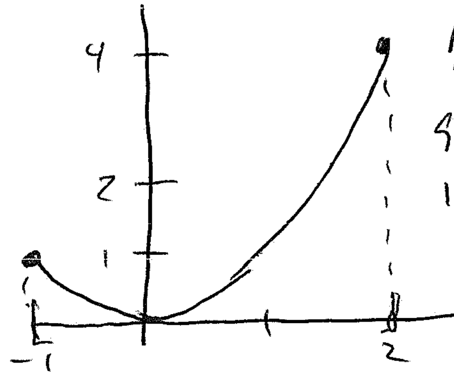
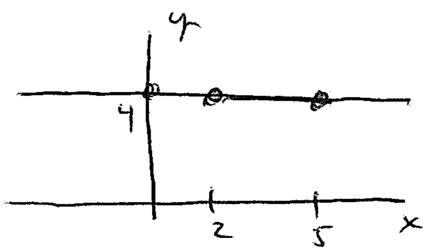
ex. Let $f(x) = x^2$ on $D = [-1,2]$. Then f has a global maximum at $x=2$ and a global minimum at $x=0$.



Here, the global max is 4 and the global min is 0

Note: f has $x=1$ as a global min but <u>No</u> global max on $(-1,2)$.

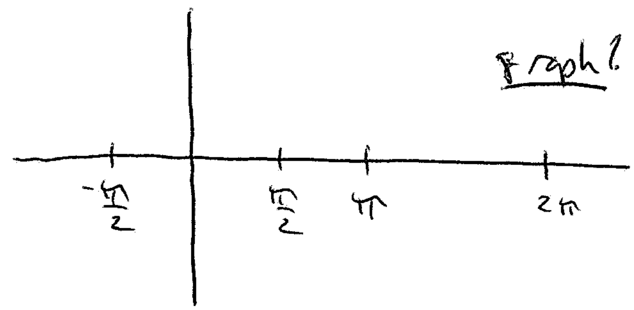
ex. $g(x) = 4$ on $D = [2, 5]$.



Where is (are) the global extrema?

ex. $h(x) = \sin x$ on $[-\frac{\pi}{2}, 2\pi]$. Where are the global extrema?

How about on $(-\frac{\pi}{2}, 2\pi)$?



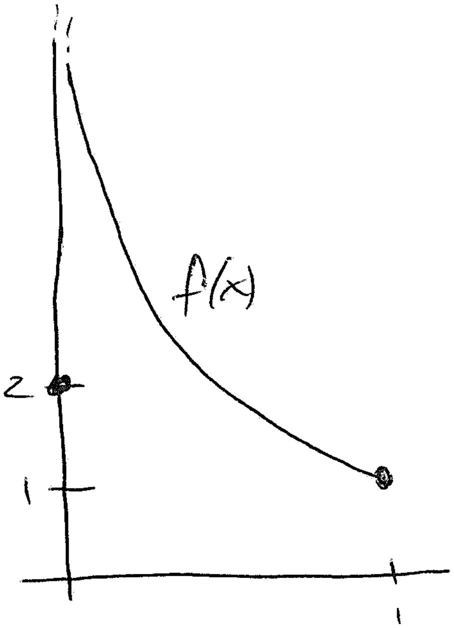
Q: Are there ways of knowing when a function will have extrema? A: Sometimes

Extreme Value Thm

If $f(x)$ is continuous on a closed finite length interval $[a,b]$, where $-\infty < a < b < \infty$, then f has a global minimum and a global max.

Section 5.1 I

Extreme Value Thm Notes



(I) $f(x)$ must be continuous.

ex. Let $f(x) = \begin{cases} \frac{1}{x} & x \in (0,1] \\ 2 & x = 0 \end{cases}$

Here f is defined on $[0,1]$ but not continuous at $x=0$. So what is the global max?

(II) Interval must be closed and bounded (finite in length).

ex. Let $f(x) = x^2$ on $[1,2]$. Global max?

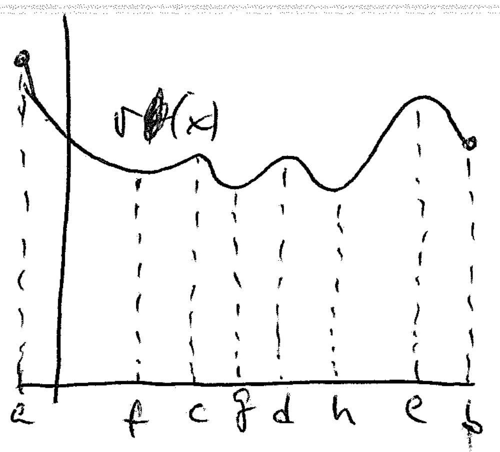
ex. $g(x) = x^2$ on $[1, \infty)$. Global max?

Def $f(x)$ on a domain D has a <u>local maximum</u> at $x=c$ if there is a $\delta>0$ so that $f(x)\le f(c)$ for all $x\in(c-\delta,c+\delta)$ in D.

Notes ① Similar def for a minimum.

② The δ does not matter. $f(c)$ just needs to be the extreme value of $f(x)$ "near" $x=c$ on both sides.

ex.



Here $r(x)$ has a global max at $x=a$ on $[a,b]$, and local max's at $x=c$, $x=d$, and $x=e$.

$r(x)$ has a global min at the smaller value of $f(x)$ at either $x=g$ or $x=h$ and local mins at $x=f$, $x=g$, $x=h$, and $x=b$.