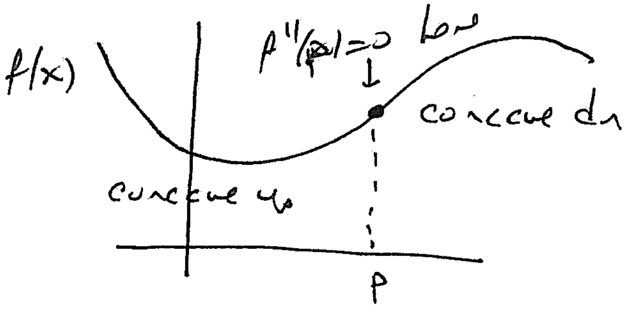
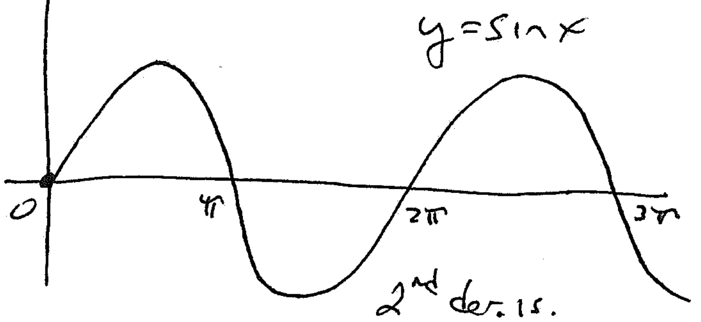
Class 25: Section 5.3. IV

Def Let $f(x)$ be continuous at a pt p.

p is called an inflection pt for f if $f''(x)$ exists and changes sign as the graph of f passes through p.

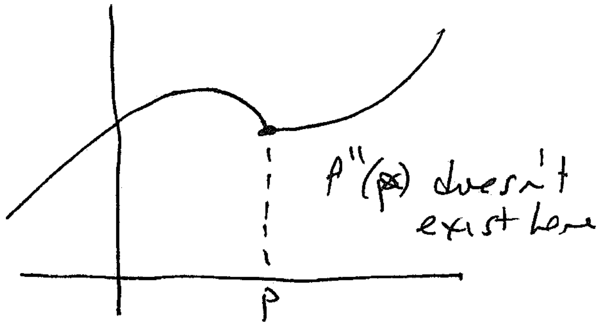


inflection pts at $k\pi$ for $k \in \mathbb{Z}$



2nd der. is $y''=-\sin x$.

$y''=0$ precisely at $x=k\pi$.



Fact if $f(x)$ is twice diff and has an inflection pt at p, then $f''(p)=0$.

ex Study the cases $f(x)=x^4$, $g(x)=-x^4$, $h(x)=x^3$.

Notice that as you pass through p, $f'(p)$ stops rising and starts falling or stops falling and starts rising. p is an inflection pt.

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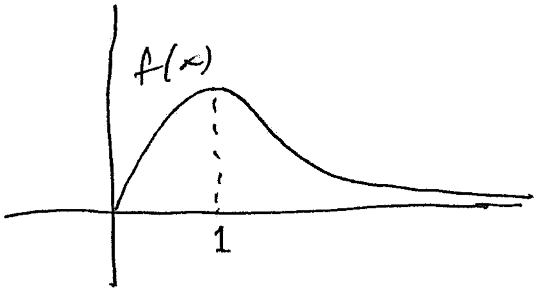
Example [redacted]:

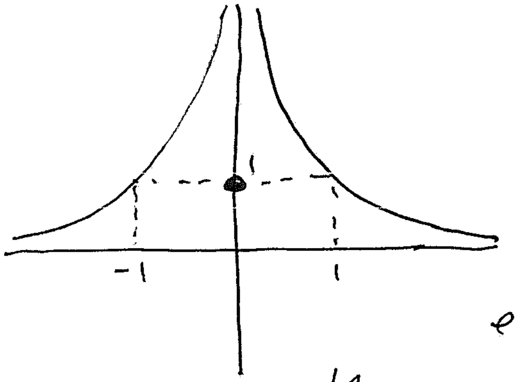
ex. For $f(x) = xe^{-x}$, find all inflection pts.



$f(x)$ is twice differentiable on all of $[0,\infty)$. Hence any inflection pts will occur where $f''(x)=0$.

Here $f''(x) = \frac{d}{dx}\left[(1-x)e^{-x}\right] = (x-2)e^{-x}$. $f''(x)=0$ only when $x=2$. And since $f''(x)<0$ on $[0,2]$ and $f''(x)>0$ on $(2,\infty)$ we conclude $x=2$ is an inflection pt.

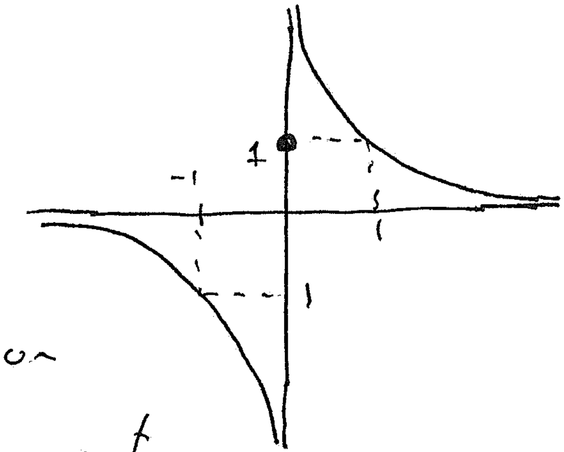
ex. Find all inflection pts for $h(x)=\begin{cases}\frac{1}{x^2} & x\neq 0\\ 1 & x=0\end{cases}$



Here $h''(x) = \frac{d}{dx}\left[-\frac{2}{x^3}\right] = \frac{6}{x^4}$.

and since $h''(x)>0$ on entire domain, except for $x=0$ there are no inflection pts.

ex. $g(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$



Here $g(x)$ is defined on the domain $\mathbb{R}$, but is *not* continuous at $x=0$ (hence $x=0$ cannot be an inflection pt). $g''(x) = \frac{2}{x^3}$ is defined on all $x \neq 0$. On $(-\infty, 0)$ $g''(x) < 0$, and on $(0, \infty)$, $g''(x) > 0$. But $x=0$ is *not* an inflection pt.

Def The horizontal line $y=b$ is called a horizontal asymptote for $f(x)$ if either

$$\lim_{x \to \infty} f(x) = b \quad \text{or} \quad \lim_{x \to -\infty} f(x) = b$$

The line $x=c$ is a vertical asymptote if either

$$\lim_{x \to c^+} f(x) = \infty \quad \text{or} \quad \lim_{x \to c^+} f(x) = -\infty$$

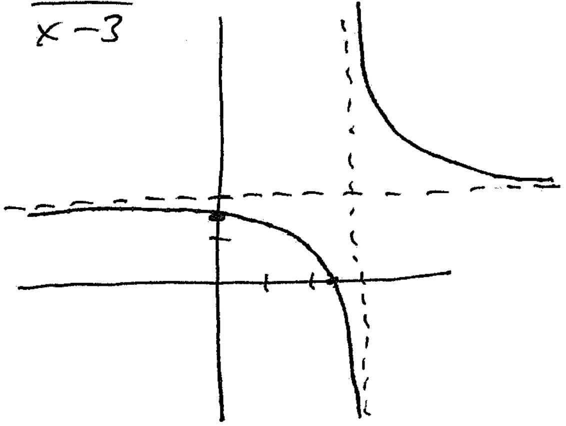
or

$$\lim_{x \to c^-} f(x) = \infty \quad \text{or} \quad \lim_{x \to c^-} f(x) = -\infty.$$

ex. Let $g(x) = \underbrace{\frac{1}{x-3} + 2} = \frac{1}{x-3} + \frac{2(x-3)}{x-3} = \frac{1+2x-6}{x-3}$

looks like $\frac{1}{x}$ shifted to the right by 3 and up by 2.

$= \frac{2x-5}{x-3}$



Here $\lim_{x\to\infty} g(x) = 2$

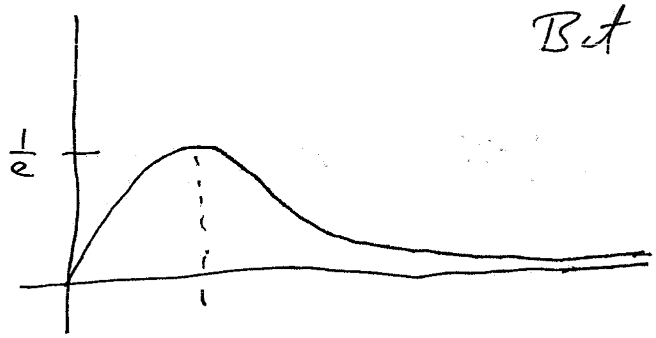
$\lim_{x\to-\infty} g(x) = 2$

$\lim_{x\to3^+} g(x) = +\infty$ and $\lim_{x\to3^-} g(x) = -\infty$.

ex. Back to $f(x) = xe^{-x}$. Does $f(x)$ have any asymptotes. Find them, if any.

<u>Solution</u>: $f(x)$ is continuous on all of $\mathbb{R}$. Hence it cannot have a vertical asymptote. (why not?).

As for horizontal asymptotes, the graph looks to have $y=0$ as a horizontal asymptote.



But $\lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty} \frac{x}{e^x}$ is hard to evaluate. We will learn a good technique soon.

There are also inclined asymptotes. These are harder to see. Example 4 pg 232 is a good example.

Note: If a function $f(x)$ can be written as a sum of 2 other functions

$$f(x) = g(x) + h(x)$$

where $\lim_{x \to \infty} h(x) = 0$, then as x gets large, $f(x)$ will look more and more like $g(x)$, even if $g(x)$ is not horizontal.

ex (4, pg 232) Let $f(x) = \frac{x^2-3}{x-2}$ on all $x \neq 2$.

We look for asymptotes and find $x=2$ to be a vertical asymptote. But $\lim_{x \to \infty} f(x) = \infty$ and $\lim_{x \to -\infty} f(x) = -\infty$ so NO horizontal asymptotes.

But there is another one!

ex (cont'd)

Here $f(x)$ is an improper rational function, and via long division, we can rewrite it:

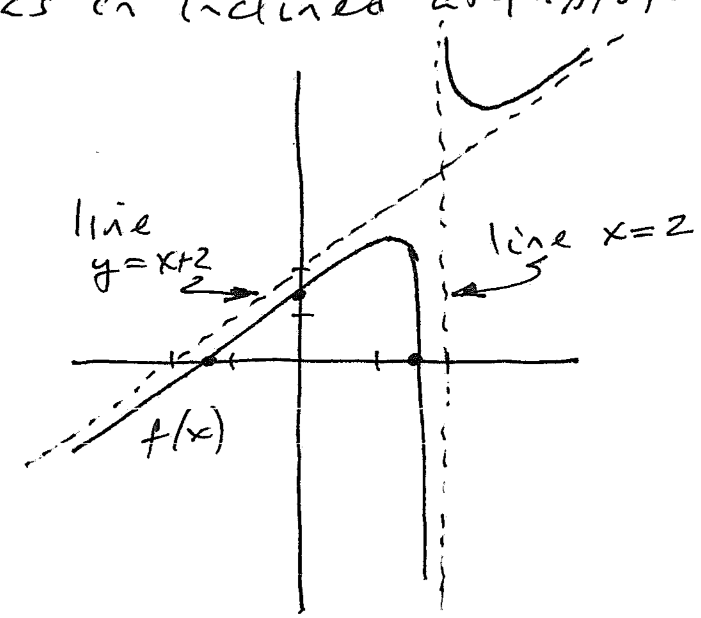
$$\begin{array}{r} x+2 \\ x-2\,\overline{)\,x^2-3} \\ \underline{x^2-2x} \\ 2x-3 \\ \underline{2x-4} \\ 1 \end{array}$$

$$f(x) = \frac{x^2-3}{x-2} = \underbrace{\underbrace{x+2}_{g(x)} + \underbrace{\frac{1}{x-2}}_{h(x)}}_{\text{from description before example began.}}$$

Hence $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} g(x) + \lim\limits_{x\to\infty} h(x)$

$= \lim\limits_{x\to\infty} g(x)$, since $\lim\limits_{x\to\infty} h(x) = 0$.

Hence when x is very large, $f(x) = \frac{x^2-2}{x-2}$ looks more and more like $g(x) = x+2$. We say $f(x)$ has an inclined asymptote at $y = x+2$:



All of the features of the graph of $f(x)$ given by calculus techniques allow us to "see" $f(x)$:

- ⓐ horizontal and vertical intercepts, if any.
- ⓑ where $f'(x)$, $f''(x)$ are positive, negative or 0.
- ⓒ local extrema
- ⓓ concavity and inflection.
- ⓔ Vertical and/or horizontal asymptotes and/or limits at ±∞.

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2 excellent examples of curve sketching in book. Here is another

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ex. Sketch $y = 3x^4 - 4x^3$.

Strategy Use list above.

Solution: Intercepts: vertical at $(0,0)$.

horizontal: $y = 0 = 3x^4 - 4x^3 = x^3(3x-4)$

Intercepts at $(0,0)$, and $(\frac{4}{3}, 0)$.

XI

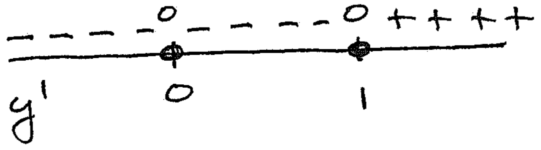
ex. Sketch $y = 3x^4 - 4x^3$ cont'd.

ⓑ Derivative info: Here

First derivative

$$\begin{cases} y' = 12x^3 - 12x^2 = 12x^2(x-1) \\ y' = 0 \text{ when } x = 0, 1. \\ y' > 0 \text{ on } (1, \infty) \\ y' < 0 \text{ on } (-\infty, 1) \end{cases}$$

schematic depicting sign of $f'(x)$.

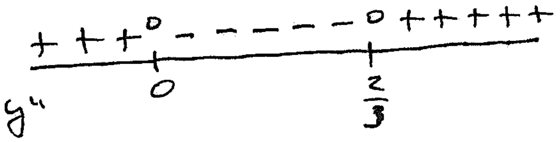


$y'' = 36x^2 - 24x = 12x(3x-2)$

$y'' = 0$ when $x = 0, \frac{2}{3}$

$y'' > 0$ on $(-\infty, 0) \cup (\frac{2}{3}, \infty)$

$y'' < 0$ on $(0, \frac{2}{3})$



ⓓ Here concave up on $(-\infty, 0) \cup (\frac{2}{3}, \infty)$ and concave down on $(0, \frac{2}{3})$.

Inflection pts at $(0,0)$ and $(\frac{2}{3}, \frac{-16}{27})$.

ⓒ $x = 1$: Here $y(1) = -1$, $y'(1) = 0$, $y''(1) > 0$

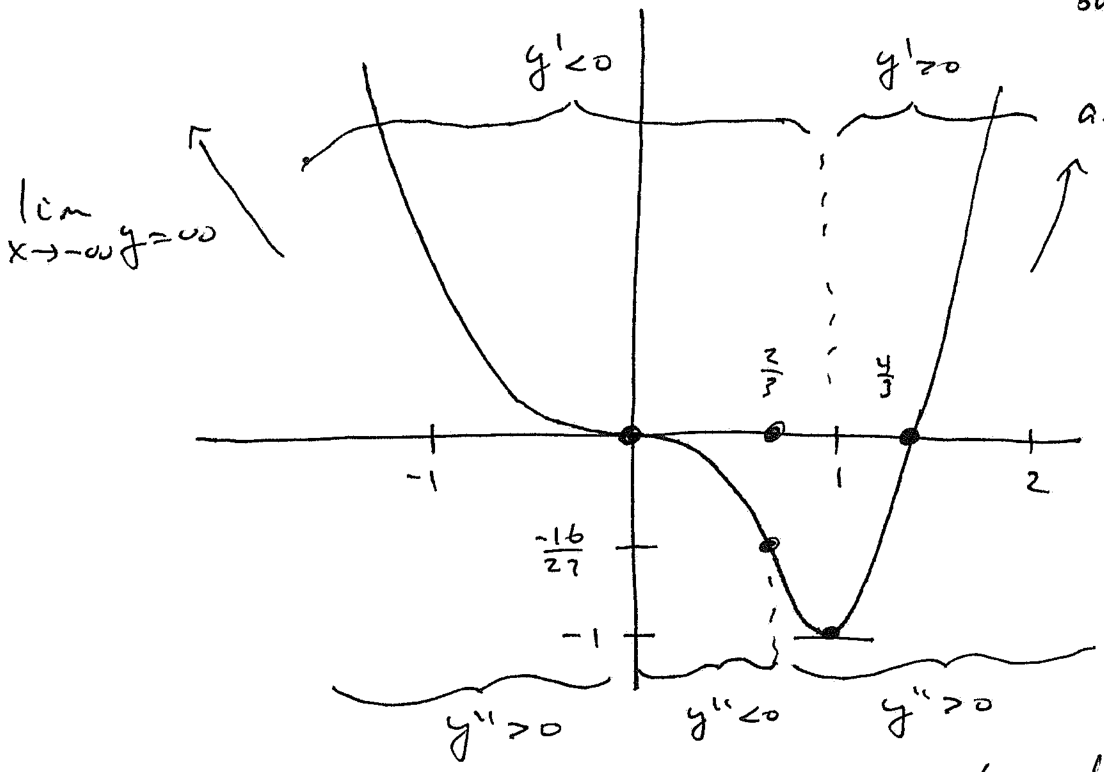
Hence there is a local min at $x = 1$ by 2nd derivative test.

$x = 0$: Here $y(0) = 0$, $y'(0) = 0$, $y''(0) = 0$ hence we do not yet know if there is an extremum at $x = 0$.

ex Sketch $y = 3x^4 - 4x^3$ cont'd.

ⓓ Since $y(x)$ is a polynomial (degree 4), there are no horizontal or vertical asymptotes.

but $\lim_{x\to\infty} y = \infty$ and $\lim_{x\to-\infty} y = \infty$. (why?)



Connecting all of the relevant information in the only way possible allows us to sketch the function.