Class 25: Section 5.4

Optimization Problems – Finding a global min or a max of a function on some interval.

<u>4 examples</u>

(I) Problem 5.6.15 The Ricker Curve of fish populations in fisheries

For what size of a fish population

$$R(P) = \alpha P e^{-\beta P}, \quad P \geq 0, \quad \alpha, \beta > 0 \text{ constants.}$$

parental stock P is the number of recruits R maximized?

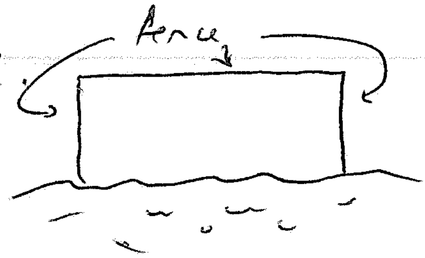
Note: For $\alpha, \beta = 1$, this is the function

$$f(x) = xe^{-x}.$$

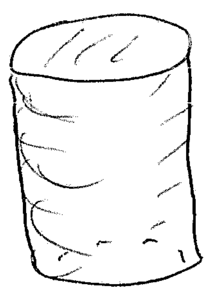
(II) Let Y be the yield of a crop as a function of nitrogen level N in the soil, modelled as $Y(N) = \frac{N}{1+N^2}$, $N \geq 0$

Find the ~~ideal~~ nitrogen level that optimizes yield.

(III) Using 1200 feet of fencing, find the dimensions of the largest enclosed rectangular area using a river as one side of the enclosure.



(IV) Minimize the amount of material required to enclose 1 liter of liquid in a closed cylindrical can.



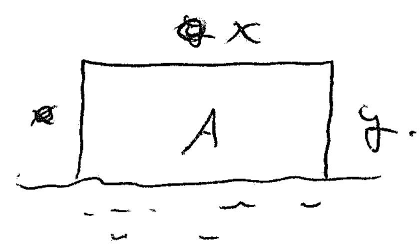
(1 liter = 1000 cm^3).

All of these problems are identical in nature:
Find the largest or smallest value of a function on some interval.

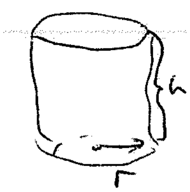
(I) Maximize $R(P) = \alpha P e^{-\beta P}$ for $P \in [0, \infty)$
For clarity, choose $\alpha = \beta = 1$.

(II) Maximize $Y(N) = \frac{N}{1+N^2}$ on $[0, \infty)$.

(III) Maximize Area $A = xy$ for fixed perimeter



(IV) Minimize Surface Area subject to fixed volume $S = \pi r^2 h + 2\pi r^2$



and $V = 1000 \text{ cm}^3$.

Technique is straightforward:

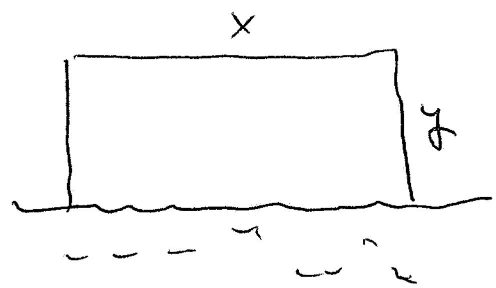
(a) Find critical pts and function values

(b) Check $\underbrace{\text{end pts}}_{\text{if interval is closed}}$ and $\underbrace{\text{limiting behavior}}_{\text{asymptotes or limits if interval is not closed.}}$

Special Note: How to do III and IV when there are two variables to play with?

Answer: When there are 2 variables, look for another relation between them to write optimized function as a function of only 1 variable.

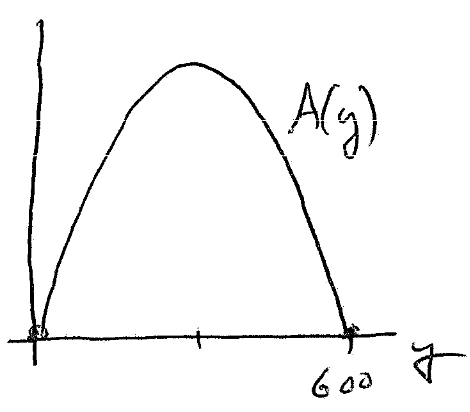
III ~~[illegible]~~ Maximize $A = xy$



where amount of fencing is fixed: Perimeter $P = 1200 = x + 2y$, or

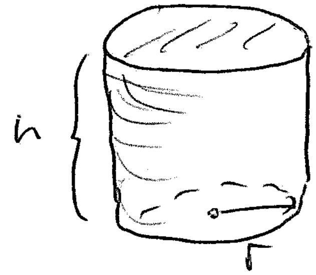
$x = 1200 - 2y$.

Hence $A = xy = (12 - 2y)y = 1200y - 2y^2$



This is now only a function of y, and $A(y)$ is a parabola opening downward.

IV Minimize $S = 2\pi r h + 2\pi r^2$, where

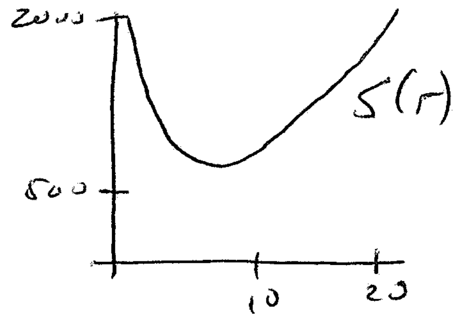


$V = 1000 = \pi r^2 h$, so $h = \frac{1000}{\pi r^2}$.

Thus $S = S(r) = 2\pi r\left(\frac{1000}{\pi r^2}\right) + 2\pi r^2$

$= \frac{2000}{r} + 2\pi r^2$

S is now only a function of r and looks like this graph →



Solutions

I Choose $\alpha = \beta = 1$. Maximize $R(p) = pe^{-p}$ on $[0,\infty)$.

Only critical pt is where $R'(p) = 0 = (1-p)e^{-p}$ at $p = 1$. And since here $R''(p) = (p-2)e^{-p}$ has $R''(1) < 0$, $p = 1$ is a local max.

And since $R(0) = 0$ and $\lim_{p\to\infty} R(p) = 0$, we conclude $R(P) = \frac{1}{e}$ is the max value of $R(p)$ on $[0,\infty)$. (we will show why soon)

II) Max $Y = \frac{N}{1+N^2}$ on $[0,\infty)$. Y here is diff on $(0,\infty)$, and $Y' = \frac{(1+N^2) - N(2N)}{(1+N^2)^2} = \frac{1-N^2}{(1+N^2)^2}$

Here $Y'(N) = 0$ only when $N = 1$.

And since $Y''(N)\Big|_{N=1} = \frac{-2N(1+N^2)^2 - (1-N^2)2(1+N^2)2N}{(1+N^2)^4}\Bigg|_{N=1}$

$= \frac{-2(4)}{16} = -\frac{1}{2} < 0$, $Y(1)$ is a local max.

Again, $Y(0) = 0$ and $\lim_{N\to\infty} Y(N) = 0$, hence $Y(1) = \frac{1}{2}$ is a global max.

III) Maximize $A(y) = 1200y - 2y^2$. This is diff on $[0, 600]$ (where $A \geq 0$), and

$A'(y) = 1200 - 4y = 0$ when $y = 300$.

Here $A''(y) = -4 < 0$ so $A(300)$ is a local max.

And since $A(0) = A(600) = 0$, $A(300)$ is a global max.

Hence $y = 300$, and thus by

Perimeter $P = 1200 = x + 2y$,

$x = 1200 - 2y = 1200 - 2(300) = 600$

Dimensions of largest enclosure are

$x = 600$ ft, $y = 300$ ft.

IV Minimize $S(r) = \frac{2000}{r} + 2\pi r^2$, diff on $(0, \infty)$. Here $S'(r) = \frac{-2000}{r^2} + 4\pi r$

and $S'(r) = 0$, where $\frac{2000}{r^2} = 4\pi r$, or

$$r = \sqrt[3]{\frac{2000}{4\pi}} \cong 5.419 \text{ cm}.$$

Here the amount of material needed can be as low as $S(5.419) \cong 369.05 + 34.05$

$\cong 403.1$ cm^2.

Check $S''(r) = \frac{4000}{r^3} + 4\pi > 0$ on $(0, \infty)$

Hence $r \cong 5.419$ is the global min.