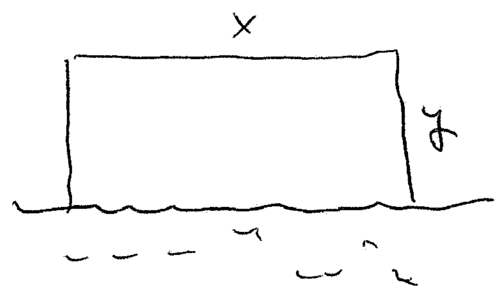
Class 27: Section 5.4

IV

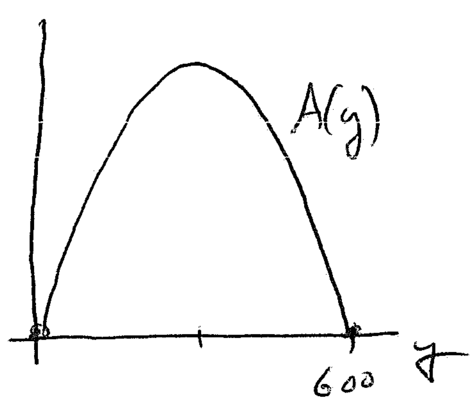
Special Note: How to do III and IV when there are two variables to play with?

Answer: When there are 2 variables, look for another relation between them to write optimized function as a function of only 1 variable.

III ~~A...~~ Maximize $A = xy$ where amount of fencing is fixed: Perimeter $P = 1200 = x + 2y$, or $x = 1200 - 2y$.



Hence $A = xy = (12-2y)y = 1200y - 2y^2$



This is now only a function of y, and A(y) is a parabola opening downward.

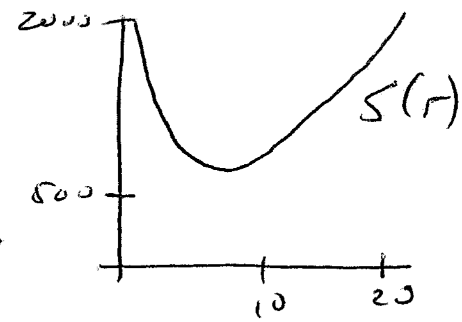
(IV) Minimize $S = 2\pi r h + 2\pi r^2$, where

$V = 1000 = \pi r^2 h$, so $h = \frac{1000}{\pi r^2}$.

Thus $S = S(r) = 2\pi r \left(\frac{1000}{\pi r^2}\right) + 2\pi r^2$

$= \frac{2000}{r} + 2\pi r^2$

S is now only a function of r and looks like this graph →



Solutions

(I) Choose $\alpha = \beta = 1$. Maximize $R(p) = pe^{-p}$ on $[0, \infty)$.

Only critical pt is where $R'(p) = 0 = (1-p)e^{-p}$ at $p = 1$. And since here $R''(p) = (p-2)e^{-p}$ has $R''(1) < 0$, $p = 1$ is a local max.

And since $R(0) = 0$ and $\lim_{p\to\infty} R(p) = 0$, we conclude $R(1) = \frac{1}{e}$ is the max value of $R(p)$ on $[0, \infty)$.

(we will show why soon)

II) Max $Y = \frac{N}{1+N^2}$ on $[0,\infty)$. Y here is diff on $(0,\infty)$, and $Y' = \frac{(1+N^2) - N(2N)}{(1+N^2)^2} = \frac{1-N^2}{(1+N^2)^2}$

Here $Y'(N) = 0$ only when $N = 1$.

And since $Y''(N)\Big|_{N=1} = \frac{-2N(1+N^2)^2 - (1-N^2)2(1+N^2)2N}{(1+N^2)^4}\Bigg|_{N=1}$

$= \frac{-2(4)}{16} = -\frac{1}{2} < 0$, $Y(1)$ is a local max.

Again, $Y(0) = 0$ and $\lim_{N \to \infty} Y(N) = 0$, hence $Y(1) = \frac{1}{2}$ is a global max.

III) Maximize $A(y) = 1200y - 2y^2$. This is diff on $[0, 600]$ (where $A \geq 0$), and

$A'(y) = 1200 - 4y = 0$ when $y = 300$.

Here $A''(y) = -4 < 0$ so $A(300)$ is a local max.

And since $A(0) = A(600) = 0$, $A(300)$ is a global max.

Hence $y = 300$, and thus by

Perimeter $P = 1200 = x + 2y$,

$x = 1200 - 2y = 1200 - 2(300) = 600$

Dimensions of largest enclosure are

$x = 600$ ft, $y = 300$ ft.

IV Minimize $S(r) = \frac{2000}{r} + 2\pi r^2$, diff on $(0, \infty)$. Here $S'(r) = \frac{-2000}{r^2} + 4\pi r$

and $S'(r) = 0$, where $\frac{2000}{r^2} = 4\pi r$, or

$$r = \sqrt[3]{\frac{2000}{4\pi}} \cong 5.419 \text{ cm.}$$

Here the amount of material needed can be as low as $S(5.419) \cong 369.05 + 34.05$

$\cong 403.1$ cm^2.

Check $S''(r) = \frac{4000}{r^3} + 4\pi > 0$ on $(0, \infty)$

Hence $r \cong 5.419$ is the global min.