Class 28: [redacted] Section 5.5 I

Back to the first optimization problem from last time: $R(p) = pe^{-p}$, or $f(x) = xe^{-x}$.

The "limiting behavior" has $R(0) = 0$ and

$$\lim_{p \to \infty} R(p) = \lim_{p \to \infty} \frac{p}{e^p}.$$

limit is not easy to "see". Cannot evaluate using Quotient Rule for limits since

$$\left.\frac{p \xrightarrow{p \to \infty} \infty}{e^p \xrightarrow[p \to \infty]{} \infty}\right\} \text{ doesn't determine anything.}$$

Why not? $\frac{\infty}{\infty}$ shows up in each of the following cases, with different results:

$$\left.\begin{aligned} \frac{x}{2x} &\xrightarrow{x \to \infty} \frac{1}{2} \\ \frac{x^2}{2x} &\xrightarrow{x \to \infty} \infty \\ \frac{x}{2x^2} &\xrightarrow{x \to \infty} 0 \end{aligned}\right\} \text{ all look like } \frac{\infty}{\infty} \text{ but ultimately have different limiting behavior.}$$

This is why we call $\frac{\infty}{\infty}$ an <u>indeterminate</u> form. ($\frac{\infty}{\infty}$ is not enough to determine the limit.)

To help better understand this indeterminate form, we study instead of where the numerator and denominator "go", but how "fast" they are going to ∞.

~~Def~~ **Thm** Let f, g be differentiable and either

Ⓐ $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$

or

Ⓑ $\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$

for $a \in \mathbb{R}$ or $a = \infty$, or $a = -\infty$.

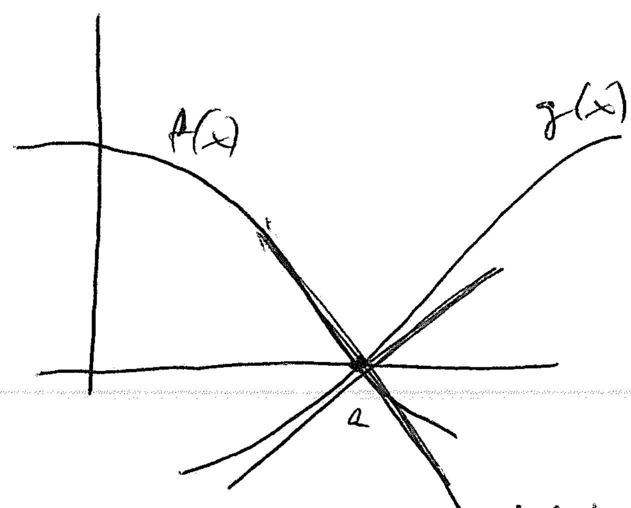
If $\lim_{x \to a} \frac{f'(x)}{g'(x)} = L$ then $\lim_{x \to a} \frac{f(x)}{g(x)} = L$.

Notes ① Called L'Hospital's Rule (L'Hôpital).

② In situation Ⓐ, a direct attempt at evaluation would yield the indeterminate form $\frac{0}{0}$. In Ⓑ, $\frac{\infty}{\infty}$

Caution: These "fractions" are only symbolic. They are not real fractions!!

③ Suppose



Here both $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$

If f, g are differentiable at a, then we can approximate f, g by tangent linear functions, and

$$f(x) \cong \cancel{f(a)}^{\,0} + f'(a)(x-a) = f'(a)(x-a)$$
$$g(x) \cong \cancel{g(a)}^{\,0} + g'(a)(x-a) = g'(a)(x-a)$$

Notes cont'd.

③ cont'd.

Then $$\frac{f(x)}{g(x)} \cong \frac{\cancel{f(c)}^{0} + f'(c)(\cancel{x-c})}{\cancel{g(c)}_{0} + g'(c)(\cancel{x-c})} = \frac{f'(c)}{g'(c)}$$

for x very close to c. So that in the limit, they will be equal.

Hence for the situation at Ⓐ and $\frac{0}{0}$,

$\lim_{x\to c} \frac{f(x)}{g(x)}$ looks like $\frac{f'(c)}{g'(c)}$ if the latter

makes sense.

④ Works with one-sided limits also.

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ex. $\lim_{x\to\infty} \frac{2x-1}{5x+3}$ Here both numerator and denominator go to ∞ as $x \to \infty$

Hence the indeterminate form $\frac{\infty}{\infty}$ as in Ⓑ.

By the Theorem,

$$\lim_{x\to\infty} \frac{2x-1}{5x+3} \overset{L'H}{=\!=} \underbrace{\lim_{x\to\infty} \frac{\frac{d}{dx}[2x-1]}{\frac{d}{dx}[5x+3]}}_{\text{if exists}} = \lim_{x\to\infty} \frac{2}{5} = \frac{2}{5}$$

ex. $\lim_{x \to \infty} \frac{2x-1}{x^2+1}$. Here direct evaluation yields another $\frac{\infty}{\infty}$ indeterminate form. Since both functions $f(x) = 2x-1$ and $g(x) = x^2+3$ are diff. L'Hospital's Rule applies, so

$$\lim_{x \to \infty} \frac{2x+1}{x^2+3} \overset{L'H}{=} \lim_{x \to \infty} \frac{2}{2x} = \lim_{x \to \infty} \frac{1}{x}$$

and this latter limit is 0.

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ex. $\lim_{x \to 0} \frac{\sin x}{x}$?

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ex $\lim_{x \to \infty} \frac{x}{e^x}$.

Other indeterminate expressions:

Hidden inside $\frac{0}{0}$ or $\frac{\infty}{\infty}$ are other forms:

$\boxed{0 \cdot \infty}$ $\quad \lim_{x \to 0^+} x \ln x$ $\quad$ Product Rule for limits results in the indeterminate $0 \cdot \infty$.

(Rewrite as $\lim_{x \to 0^+} \frac{\ln x}{1/x}$ to get indeterminate $\frac{\infty}{\infty}$)

$\boxed{\infty - \infty}$: $\lim_{x\to 0^+} \cot x - \csc x$

Note: Each of $\cot x$ and $\csc x$ have vertical asymptotes at $x=0$. Hence direct eval. yields $\infty - \infty$.

Rewrite $\lim_{x\to 0^+} \frac{\cos x}{\sin x} - \frac{1}{\sin x} = \lim_{x\to 0^+} \frac{\cos x - 1}{\sin x}$

$$\underset{\frac{0}{0}}{\overset{L'H}{=}} \lim_{x\to 0^+} \frac{-\sin x}{\cos x} = \lim_{x\to 0^+} -\tan x = 0.$$

Also, $[f(x)]^{g(x)}$ may be problematic:

$$\left.\begin{aligned} 0^0 &: \lim_{x\to 0^+} x^x \\ \infty^0 &: \lim_{x\to 0^+} \left(\tfrac{1}{x}\right)^x \\ 0^\infty &: \lim_{x\to 0^+} x^{1/x} \\ 1^\infty &: \lim_{x\to 0^+} (\cos x)^{1/x} \end{aligned}\right\}$$

All of these involve competing quantities

ex. $$\lim_{x\to 0^+} (\cos x)^{1/x} = \lim_{x\to 0^+} \exp\left[\ln(\cos x)^{1/x}\right]$$
$$= \lim_{x\to 0^+} \exp\left[\tfrac{1}{x}\ln(\cos x)\right]$$
$$= \exp\left[\lim_{x\to 0^+} \frac{\ln(\cos x)}{x}\right].$$
$$\underset{\frac{0}{0}}{\overset{L'H}{=}} \exp\left[\lim_{x\to 0^+} \frac{\frac{1}{\cos x}(-\sin x)}{1}\right] = \exp\left[\lim_{x\to 0^+} -\tan x\right] = \exp[0]$$
$$= 1.$$

Experiment:

$$\lim_{x\to\infty} \frac{x}{e^x} \stackrel{L'H}{=\!=} \lim_{x\to\infty} \frac{1}{e^x} = 0$$

$$\lim_{x\to\infty} \frac{x^2}{e^x} \stackrel{L'H}{=\!=} \lim_{x\to\infty} \frac{2x}{e^x} \stackrel{L'H}{=\!=} \lim_{x\to\infty} \frac{2}{e^x} = 0$$

$$\lim_{x\to\infty} \frac{x^3}{e^x} \stackrel{L'H}{=\!=} \lim_{x\to\infty} \frac{3x^2}{e^x} \stackrel{L'H}{=\!=} \lim_{x\to\infty} \frac{6x}{e^x} \stackrel{L'H}{=\!=} \lim_{x\to\infty} \frac{6}{e^x} = 0$$

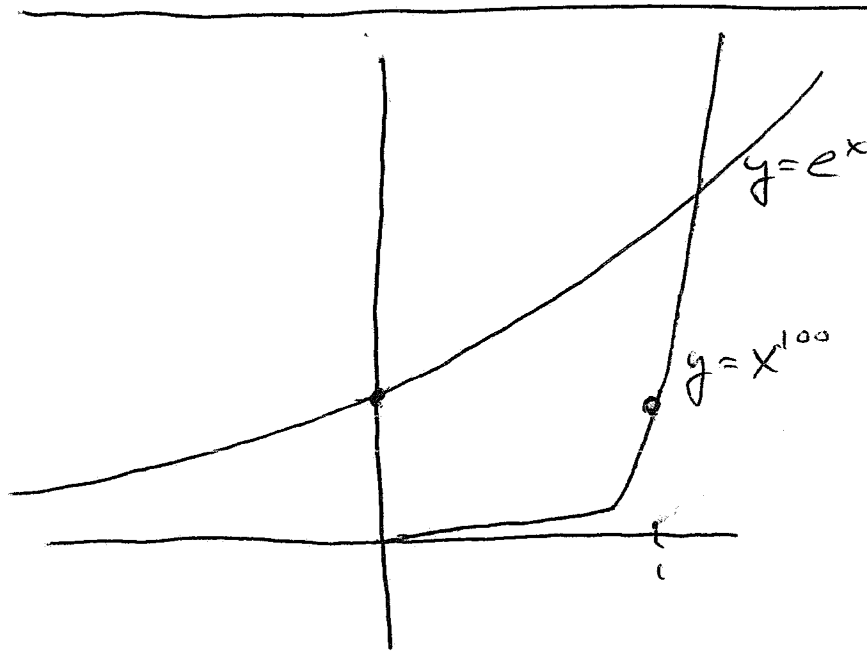
$\vdots$

$$\lim_{x\to\infty} \frac{x^{400}}{e^x} = ?$$

$\vdots$

$$\lim_{x\to\infty} \frac{x^p}{e^x} = ? \quad \text{for } p \in \mathbb{N}.$$

Show $\displaystyle\lim_{x\to\infty} \frac{x^p}{e^x} = p!\left(\lim_{x\to\infty} \frac{1}{e^x}\right) = (p!)(0) = 0.$



it looks like x^{100} will approach ∞ faster than e^x. But

$$\lim_{x\to\infty} \frac{x^{100}}{e^x} = 0.$$