# Class 29: Section 5.8

The idea of calculating a derivative of a differentiable function $f(x)$ in order to study the properties of $f$ is clear.

But there are many situations where we know how a function is changing but we do not know the function

ex. Newtons $2^{nd}$ Law of Motion

$$F = ma$$

ex. Find a function whose derivative is $\tan x$:

Solve $\frac{dy}{dx} = \tan x$ for $y(x)$.

This involves going backwards from the act of differentiating back to the original function from the derivative.

This is like in general solving $\frac{dy}{dx} = f(x)$ for some unknown function $y(x)$.

ex. $y' = \cos x$. This has solution $y = \sin x$ but also $y = 3 + \sin x$.

ex. $y' = \tan x$ has solution $y = -\ln|\cos x|$. Any other solutions?

ex. $y' = \frac{dy}{dx} = x^n$, for $n \in \mathbb{R}$, $n \neq -1$, and

$y' = x^{-1}$

Def A function $F(x)$ is called an antiderivative of $f(x)$ on an interval $I$ if $F'(x) = f(x)$ for all $x \in I$.

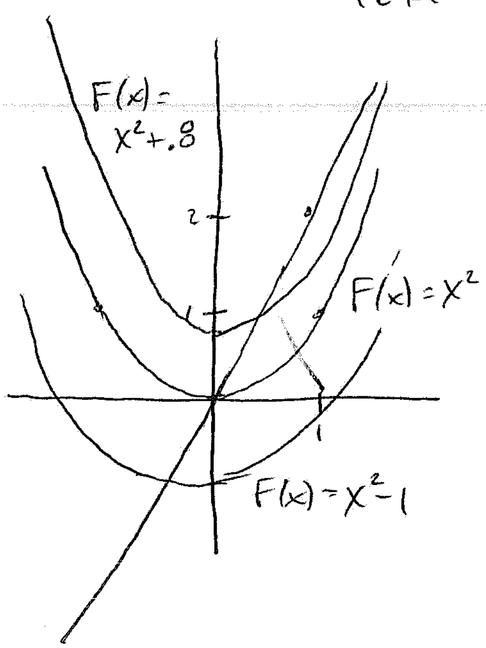
Notes ① There is no general procedure for finding antiderivatives. Most of it is pattern recognition. There are a couple of "rules" for certain situations.

Notes cont'd

② If $F(x)$ is an antiderivative of $f(x)$, then so is $F(x)+C$ for any choice of $C \in \mathbb{R}$ (why?) Thus there are always many antiderivatives of a function, if it has any. But

Fact: Any 2 antiderivatives of $f(x)$ differ by a constant.

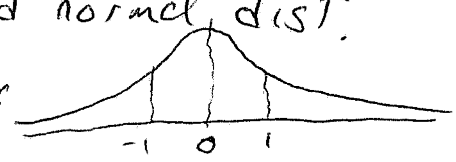
ex. All antiderivatives of $f(x) = 2x$ look like $F(x) = x^2 + C$ for some choice of $C \in \mathbb{R}$.




③ Some functions do not have "easy to write" antiderivatives.

ex. No easy expression for $F(x)$ when $f(x) = e^{-x^2}$

(this function shows up in the standard normal dist. in statistics




function is $Ce^{-\frac{(x-\mu)^2}{2\sigma^2}}$

Let $\mu = 0$ $\sigma = 1/\sqrt{2}$

Notes cont'd

④ Finding the "general" form for the antiderivative of a function involves finding any one $F(x)$ and then adding an unknown constant (typically denoted by a capital "$C$")

ex. For $g(x) = 2x+1$, the general antiderivative is
$$G(x) = x^2 + x + C$$

ex. The "general solution" (a form for all solutions) to $\frac{dy}{dx} = e^{ax}$, $\underset{a \neq 0}{a \in \mathbb{R}}$, is $y(x) = \frac{1}{a}e^{ax} + C$

⑤ If we knew one pt of an antiderivative, we can use that information to solve for the value of $C$. In this case, there would be only 1 antiderivative.

Notes cont'd

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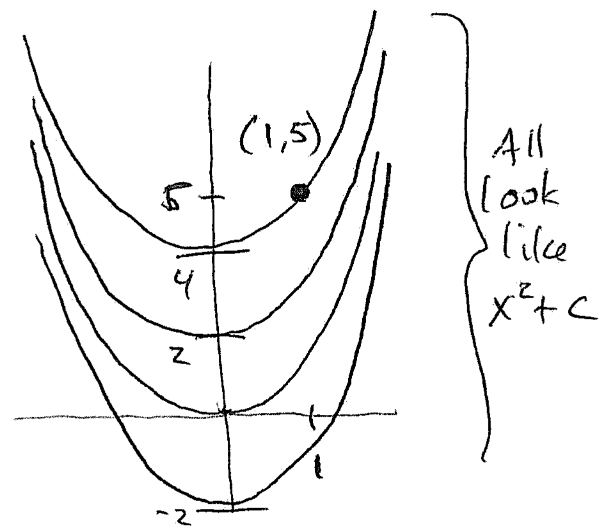
ex Find the antiderivative of $f(x) = 2x$ that satisfies $F(1) = 5$.




Solution: The general antiderivative is $F(x) = x^2 + c$. And if $F(1) = (1)^2 + c = 5$, then $c = 4$. Hence the antiderivative of $f(x) = 2x$ that passes through $(1,5)$ is

$$F(x) = x^2 + 4$$

ex. Solve for the unknown function $y(x)$ where

$$\frac{dy}{dx} = e^{3x}, \text{ and } y(0) = -1$$

Solution: The general antiderivative of $e^{3x}$ is $y(x) = \frac{1}{3}e^{3x} + c$. And if $y(0) = -1$, then $y(0) = \frac{1}{3}e^{3(0)} + c = \frac{1}{3} + c = -1$, so $c = \frac{-4}{3}$.

So $y(x) = \frac{1}{3}e^{3x} - \frac{4}{3}$.

Def. ① An equation involving an unknown function $y(x)$ and some of its derivatives is called an ordinary differential equation, or ODE.

② An ODE with some data (like a pt on the antiderivative) is called an initial value problem, or IVP.

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Some Antiderivative Patterns

| Function $f(x)$ | Antiderivative $F(x)$ |
|---|---|
| $x^n,\ n\neq -1$ | $\frac{1}{n+1}x^{n+1}$ |
| $e^{ax},\ a\neq 0$ | $\frac{1}{a}e^{ax}$ |
| $\sin(ax)$ | $-\frac{1}{a}\cos(ax)$ |
| $\frac{1}{x}$ (this is the case $x^n,\ n=-1$) | $\ln\lvert x\rvert$ (absolute values here) |

For the last one, let $f(x) = \ln x$, for $x > 0$.

Then $f'(x) = \frac{1}{x}$ but only on $(0, \infty)$.

Let $g(x) = \ln(-x)$, defined on $(-\infty, 0)$. Then

$$g'(x) = \frac{d}{dx}\left[\ln(-x)\right] = \frac{1}{(-x)} \cdot \frac{d}{dx}[-x] = \frac{1}{(-x)} \cdot (-1) = \frac{1}{x}$$

but the domain is only on $(-\infty, 0)$.

Put these together:

$$h(x) = \ln|x| = \begin{cases} \ln x & x > 0 \\ \ln(-x) & x < 0 \end{cases} \quad \text{on } x \neq 0$$

Then $h'(x) = \frac{1}{x}$ on all $x \neq 0$.