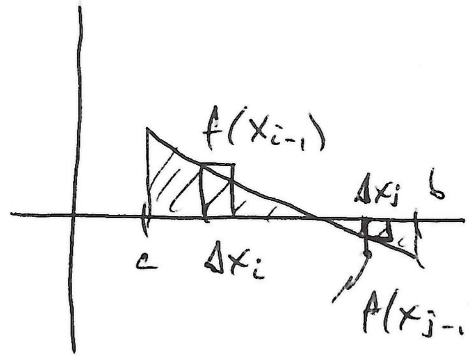
Class 31: Section 6.2

Q: What happens in the definition of a definite integral when the function dips below the x-axis?



Here

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

and the boxes where the function values are negative will result in "negative areas". This is fine and we treat them as such:

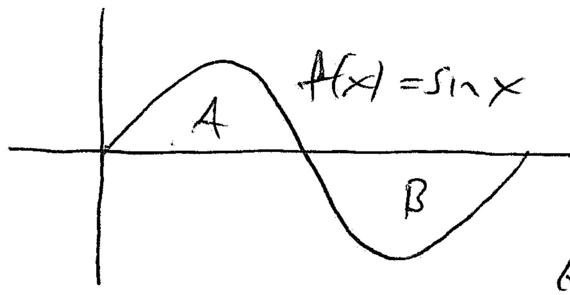
<u>Fact</u> If $f(x)$ is integrable on $[a,b]$ and $f(x) \geq 0$ on $[a,b]$, then

$$\int_a^b f(x)\,dx = \begin{pmatrix} \text{area between } f(x) \text{ and} \\ x\text{-axis on } [a,b] \end{pmatrix}$$

Otherwise, for $f(x)$ integrable,

$$\int_a^b f(x)\,dx = (\text{area above } x\text{-axis}) - (\text{area below } x\text{-axis})$$

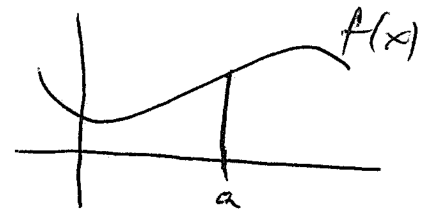
ex. It is known that the 2 lobes in the sine function, A and B are of the same area.



Hence $\int_0^{2\pi} \sin x\,dx = 0$.

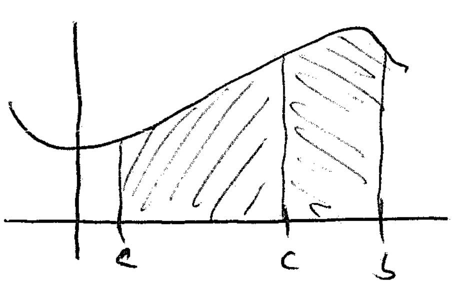
Other properties of integrals

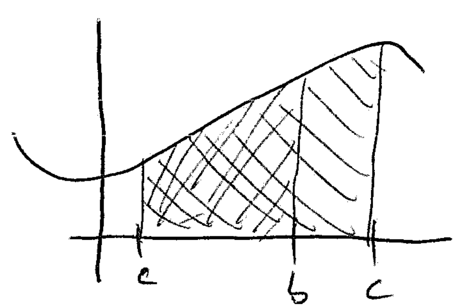
① $\int_a^a f(x)\,dx = 0$ why?



ⓑ $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ (integrating backwards results in negative values for each Δx_i).

ⓒ $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ for any interval containing a, b, c.





why does this work also?

(d) $\displaystyle\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$

(e) $\displaystyle\int_a^b \big(f(x)+g(x)\big)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

(f) If $f(x) \geq 0$ on $[a,b]$, then $\displaystyle\int_a^b f(x)\,dx \geq 0$ also.

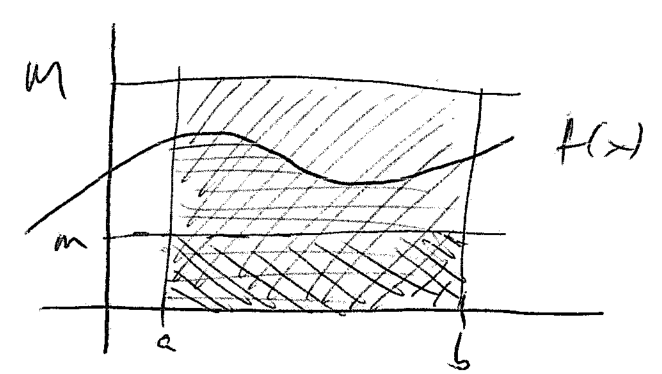
(g) ~~[illegible]~~. If $f(x) \leq g(x)$ on $[a,b]$, then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$

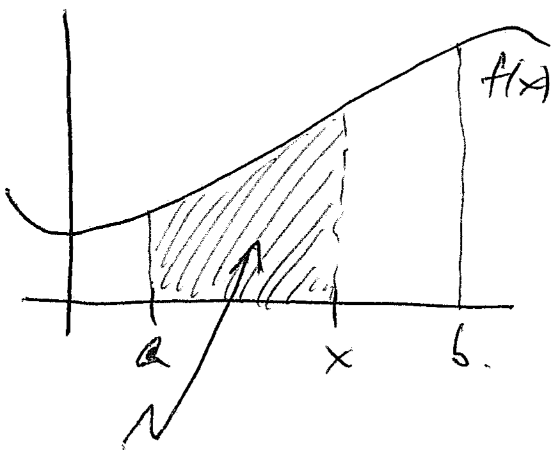
even if both are <u>negative</u>.

(h) If $m \leq f(x) \leq M$ on $[a,b]$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$



Lets create a new function: Let $f(x)$ be continuous on $[a,b]$, and for any $x \in [a,b]$, let



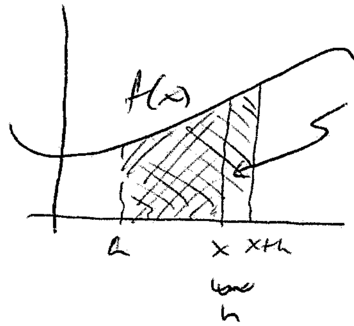
$$F(x) = \int_a^x f(u)\,du$$

value of $F(x)$ is area of this region.

its easy to see that this function is continuous (move x slightly, and the area changes slightly). Is $F(x)$ differentiable? To see, calculate $F'(x)$.

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} \frac{\int_a^{x+h} f(u)\,du - \int_a^x f(u)\,du}{h}$$

$$= \lim_{h \to 0} \frac{\left(\int_x^{x+h} f(u)\,du\right)}{h},$$



$\int_x^{x+h} f(u)\,du$

Now when h is very small,

$$\int_x^{x+h} f(u)\,du \approx f(x) \cdot h$$

Hence $F'(x) = \lim_{h \to 0} \dfrac{\left(\int_x^{x+h} f(u)\,du\right)}{h} = \lim_{h \to 0} \dfrac{f(x)h}{h} = f(x).$

Hence $\frac{d}{dx}[F(x)] = f(x)$.

Fundamental Thm of Calc (I)

If $f(x)$ is cont on $[a,b]$, then

$$F(x) = \int_a^x f(u)\,du, \quad x \in [a,b],$$

is continuous on $[a,b]$, differentiable on (a,b), and

$$F'(x) = f(x).$$

<u>Notes</u> ① Crazy but true, so that

$$\frac{d}{dx}\left(\int_a^x f(u)\,du\right) = f(x)$$

One can easily "see". Also, this means that Integrals and derivatives are inverses of each other.

ex. $\frac{d}{dx}\left(\int_0^x (4t^2 - \tan\sqrt{t^2+1} + e^t)\,dt\right) = 4x^2 - \tan\sqrt{x^2+1} + e^x$

② This means that ~~the~~ an antiderivative of $f(x)$ can be written like $F(x)$:

Antiderivatives of $f(x)$ look like $\int_a^x f(u)\,du$

The general antiderivative of $f(x)$ is then

$$C + \int_a^x f(u)\,du$$

Def. For a continuous function $f(x)$, a the general antiderivative is written

$$\int f(x)\,dx$$

(without limits).

ex. For $f(x) = \cos x$,

$$\int \cos x \, dx = \sin x + C.$$

ex. $\int 2x \, dx = x^2 + C$

ex. $\int e^{kx} dx = \frac{1}{k} e^{kx} + C.$

ex. $\int x^n dx = \frac{1}{n+1} x^{n+1} + C$ for $n \in \mathbb{R}$, $n \neq -1$.

ex $\int \frac{1}{x} dx = \ln|x| + C.$

ex. $\int \sec^2 x \, dx = \tan x + C.$

Fundamental Thm of Calculus (II).

Go back to $F(x) = \int_b^x f(u) du$. Here $F(x)$ plays the role of an antiderivative of $f(x)$.

16 $G(x) = \int_a^x f(u)\,du$ were another antiderivative of $f(x)$, then

$$G(x) = F(x) + C.$$

But then $G(a) = \int_a^a f(u)\,du = 0 = F(a) + C.$

And since $F(a) = \int_b^a f(u)\,du$, this implies $C =$

$$-\int_b^a f(u)\,du = \int_a^b f(u)\,du = -F(a).$$

Thus $G(x) = F(x) - F(a)$, or

$$\int_a^x f(u)\,du = F(x) - F(a)$$

Setting $x=b$, we get $\int_a^b f(u)\,du = F(b) - F(a)$

FTC (II)

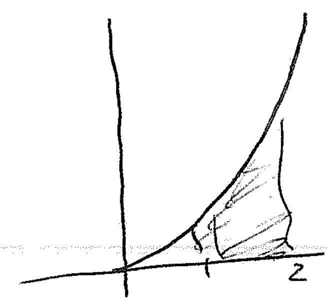
For $f(x)$ continuous on $[a,b]$,

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b.$$

where $F(x)$ is any antiderivative of $f(x)$.

Notes: ① Now, once an antiderivative is known for $f(x)$, any definite integral of $f(x)$ can be computed

ex. $\int_1^2 x^2\,dx = ?$



Here $f(x) = x^2$, $F(x) = \frac{1}{3}x^3$.

Hence $\int_1^2 x^2\,dx = \frac{1}{3}x^3\Big|_1^2 = \frac{1}{3}(2)^3 - \frac{1}{3}(1) = \frac{8}{3} - \frac{1}{3} =$

Compare this to some estimates

ex. $\int_0^{2\pi} \sin x\,dx\,?$ For $f(x) = \sin x$, $F(x) = -\cos x$.

$= -\cos x\Big|_0^{2\pi} = -\cos(2\pi) + \cos(0) = -(1) + 1 =$