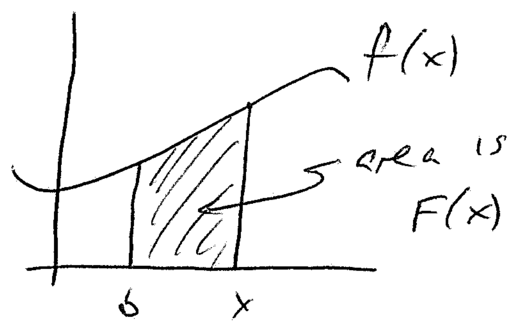
Class 32 Section 6.3

Fundamental Theorem of Calculus (II)

Go back to $F(x)=\int_b^x f(u)\,du$. Here, $F(x)$ plays the role of an antiderivative of $f(x)$.



Now, let $G(x)=\int_a^x f(u)\,du$ be a different antiderivative of $f(x)$.

Since it is another antiderivative, it must differ from $F(x)$ only by a constant, so

$$G(x)=F(x)+C$$

for some real number $C\in\mathbb{R}$.

But then, evaluated at $x=a$, we have

$$G(a)=\int_a^a f(u)\,du=0=F(a)+C$$

We also know by the definition of $F(x)$, that

$$F(a) = \int_b^a f(u)\,du, \quad \text{so that}$$

$$C = -\int_b^a f(u)\,du = \int_a^b f(u)\,du = -F(a)$$

Thus, we see that $G(x) = F(x) \underbrace{- F(a)}_{\text{this was } C}$,

or

$$\int_a^x f(u)\,du = F(x) - F(a)$$

Finally, evaluate this at $x = b$ to get

$$\boxed{\int_a^b f(u)\,du = F(b) - F(a)}$$

This relates directly the antiderivatives of $f(x)$ to the definite integral of $f(x)$ on the interval $[a,b]$.

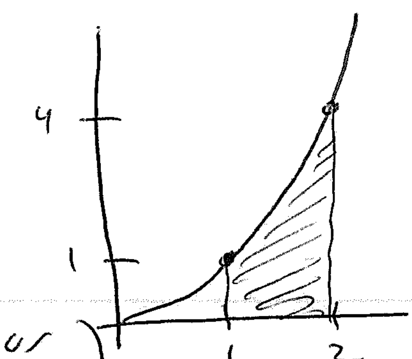
Fundamental Theorem of Calculus (II)

If $f(x)$ is continuous on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b,$$

where $F(x)$ is <u>any</u> antiderivative of $f(x)$.

<u>Notes</u> ① Now, once <u>an</u> antiderivative of $f(x)$ is known, the definite integral of $f(x)$ can be computed!



ex. Calculate $\int_1^2 x^2\,dx$.

Soln: Let $f(x) = x^2$. Then

$F(x) = \frac{x^3}{3} + C$ (choose your favorite $C \in \mathbb{R}$)

Then $\int_1^2 x^2\,dx = \left(\frac{x^3}{3} + c\right)\Big|_1^2 = \left(\frac{(2)^3}{3} + c\right) - \left(\frac{(1)^3}{3} + c\right)$

$= \frac{8}{3} + c - \frac{1}{3} - c = \frac{7}{3}$.

② Notice that whatever C you chose, it always cancels out in the end. Might as well <u>choose</u> $C = 0$.

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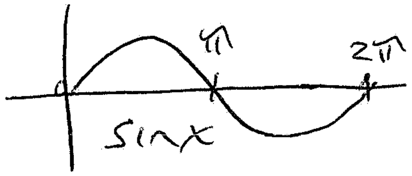
ex



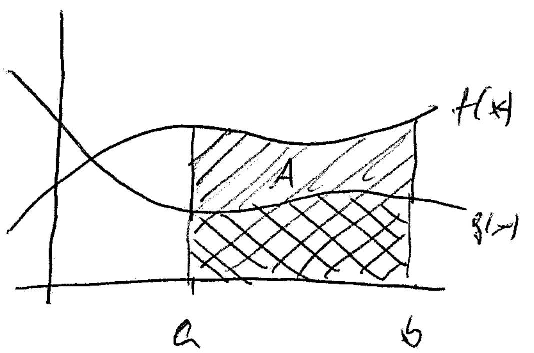
$\int_0^{2\pi} \sin x\,dx = (-\cos x)\Big|_0^{2\pi} = (-\cos 2\pi) - (-\cos 0)$

$= (-1) - (-1) = 0$ <u>as before</u>.

Some applications of the definite integral

(I) Calculating areas

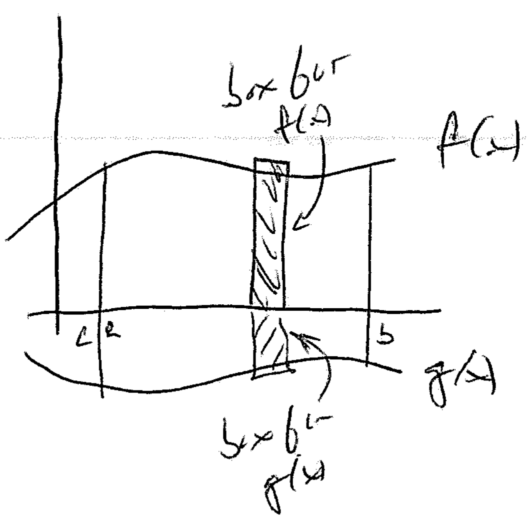
Suppose $f(x) \geq g(x) \geq 0$ on $[a,b]$. Then the area of A is



$$\text{area}(A) = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

$$= \int_a^b \big(f(x) - g(x)\big)\,dx$$

But this works even if $g(x) < 0$ on $[a,b]$.

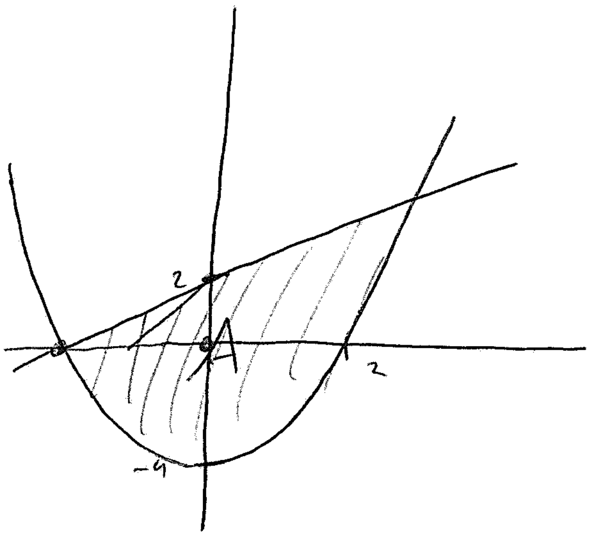


$$\text{area}(A) = \int_a^b f(x)\,dx - \underbrace{\int_a^b g(x)\,dx}_{\substack{\text{already} < 0 \\ \text{since } g(x) < 0.}}$$

with the second minus sign, the contribution of $g(x)$ is counted as positive.

No matter the signs of $f(x)$ and $g(x)$, as long as $f(x) \geq g(x)$ on $[a,b]$, the formula holds!

ex. Find the area of the region bounded by the curves $y = x^2 - 4$, $y = x + 2$.



Strategy: Locate where graphs cross and integrate using the intersections as limits.

Solution: Here.

$$x^2 - 4 = y = x + 2.$$

$$x^2 - x - 6 = 0.$$

where $x^2 - x - 6 = (x-3)(x+2)$. Thus the 2 functions intersect at $x = -2$ and $x = 3$.

On the interval $[-2, 3]$, $f(x) = x + 2 > g(x) = x^2 - 4$,

so $\text{area}(A) = \int_{-2}^{3} (f(x) - g(x))\,dx = \int_{-2}^{3} (x + 2 - x^2 + 4)\,dx$

$$= \int_{-2}^{3} (-x^2 + x + 6)\,dx$$

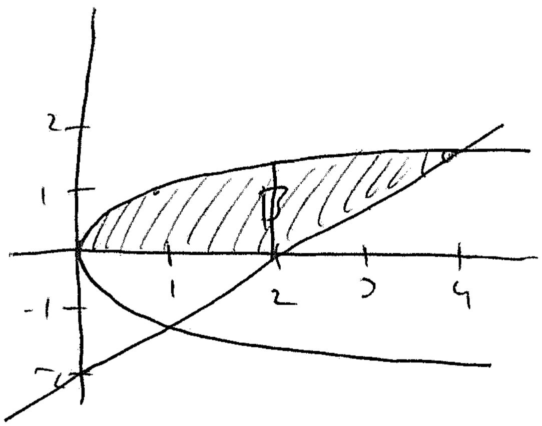
$$= \left(-\frac{x^3}{3} + \frac{x^2}{2} + 6x\right)\Bigg|_{-2}^{3}$$

$$= \frac{-27}{3} + \frac{9}{2} + 18 - \left(-\frac{(-2)^3}{3} + \frac{(-2)^2}{2} + 6(-2)\right)$$

$$= -9 + \frac{9}{2} + 18 - \frac{8}{3} - 2 + 12 = 19 + \frac{9}{2} - \frac{8}{3} = 20\frac{5}{6}.$$

Also, exercise 3 pg 309 is a good one:

ex. Find the ~~region~~ area of the region bounded by $y=\sqrt{x}$ and $y=x-2$ in the first quadrant.



Strategy 1: Divide into 2 regions and integrate with $f(x)=\sqrt{x}$ and $g(x)=x-2$.

$$\text{area}(B) = \int_0^2 \sqrt{x}\,dx + \int_2^4 \left(\sqrt{x}-(x-2)\right)dx.$$

Make sense? So

$$\text{area}(B) = \int_0^2 \sqrt{x}\,dx + \int_2^4 (\sqrt{x}-x+2)\,dx$$

$$= \frac{2}{3}x^{3/2}\Big|_0^2 + \left(\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right)\Big|_2^4$$

$$= \left(\frac{2}{3}\sqrt{8} - 0\right) + \left(\frac{2}{3}(8) - \underbrace{8 + 2(4)}_{0} - \left(\frac{2}{3}\sqrt{8} - 2 + 4\right)\right)$$

$$= \frac{16}{3} - 2 = \frac{10}{3}.$$

Strategy 2: Write as functions of y and integrate from 0 to 2: