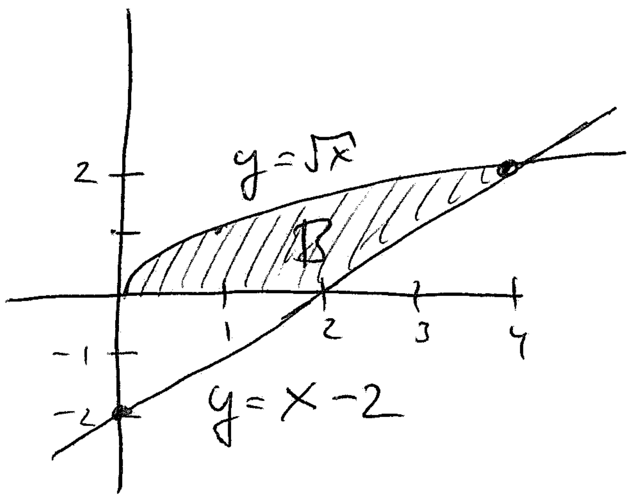
Class 33: Section 6.3



ex Calculate the area of the shaded region.

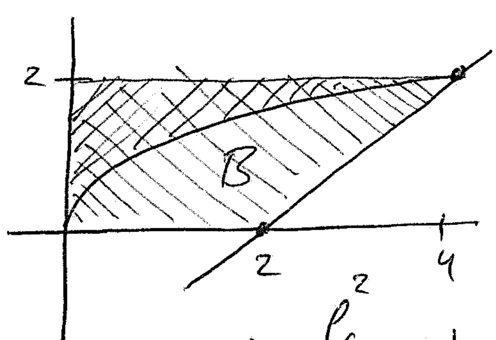
Strategy: Rewrite both functions as functions of y, on $y \geq 0$, then integrate with respect to y, with $f(y) = y+2$ and $g(y) = y^2$, on $[0,2]$.

$$\text{Then area}(B) = \int_0^2 (f(y) - g(y))\,dy = \int_0^2 (y+2-y^2)\,dy$$

$$= \left(\frac{y^2}{2} + 2y - \frac{y^3}{3}\right)\Bigg|_0^2 = \left(\frac{(2)^2}{2} + 2(2) - \frac{(2)^3}{3}\right) - 0$$

$$= 2 + 4 - \frac{8}{3} = 6 - \frac{8}{3} = \frac{18}{3} - \frac{8}{3} = \frac{10}{3}.$$

In essence, we are calculating area between a function of y and the y-axis.



$$\text{area}(B) = \int_0^2 (y+2)\,dy - \int_0^2 y^2\,dy.$$

(II) Cumulative change

Given only the derivative of some function, one can recover the function via integration.

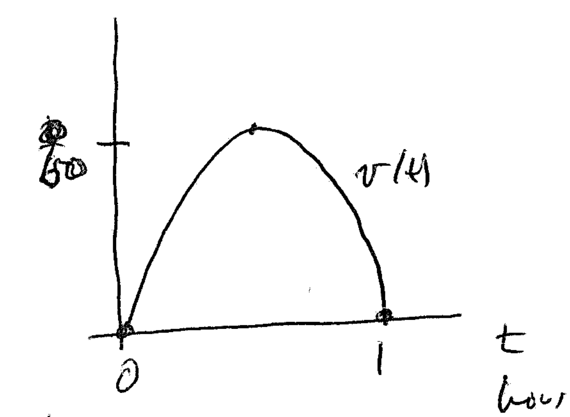
Over an interval, this is like "adding" up all of the instantaneous changes in a function.

The result is the total change in the function

ex. Suppose you track your velocity on your speedometer and arrive at $v(t) = 120t(1-t)$ in your hour drive.

How far have you gone?

$$\int_0^1 v(t)\,dt = s(1) - s(0)$$



by the FTC, where position is $s(t)$, and $s'(t) = v(t)$.

$$\int_0^1 v(t)\,dt = \int_0^1 120\,t(1-t)\,dt = \int_0^1 (120t - 120t^2)\,dt$$

$$= (60t^2 - 40t^3)\Big|_0^1 = 60 - 40 = 20 \text{ meters.}$$

~~Hence~~ Put another way, we can say that

$$f(b) - f(a) = \int_a^b f'(x)\,dx$$

so that if we integrate the rate of change of a function over an interval, we recover the total change of that function over the interval.

III The average value of a function

The average value of a discrete, finite set of numbers is the mean:

$$\text{average value is } \frac{\sum_{i=1}^{n} a_i}{n}.$$

But what is the average value of a continuous function over an interval?

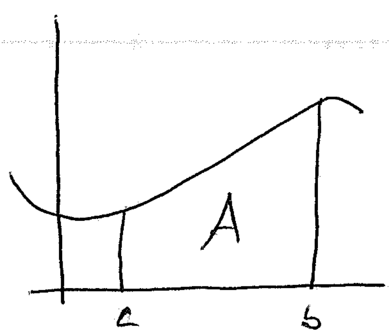
Since an integral is like adding, we can still define it as the total sum of the function values divided by the inputs

In this case:

$$f_{avg} = \begin{matrix}\text{average value of } f \\ \text{on } [a,b]\end{matrix} = \frac{\int_a^b f(x)\,dx}{b-a} = \frac{\text{total sum of fnc values}}{\text{length of interval}}$$

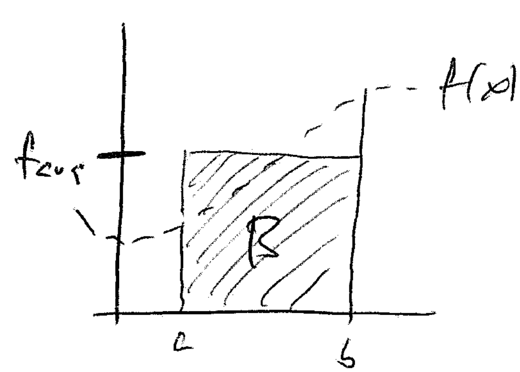
$$= \frac{1}{b-a}\int_a^b f(x)\,dx$$

Geometric interpretation?



Think of the block under $f(x)$ from a to b as the snapshot of a fluid.

Allow the fluid to settle, and the fluid height is the average value.



area(A) = area(B)

$\int_a^b f(x)\,dx = (f_{avg})(b-a)$

ex. #25 pg 323.

The average value of $f(x) = x^2 - 2$ on $[0,2]$

" $f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$

$= \frac{1}{2-0}\int_0^2 (x^2-2)\,dx = \frac{1}{2}\left[\left(\frac{x^3}{3} - 2x\right)\Big|_0^2\right]$

$= \frac{1}{2}\left(\frac{8}{3} - 4 - (0-0)\right) = -\frac{2}{3}$