Class 34: Section 7.1

Differentiation, in general, is a fairly mechanical process, with the ~~elementary~~ derivatives of the elementary functions known and a small set of rules to handle the more complicated combinations and compositions of functions.

Integration, on the other hand, is not so straightforward. There are some rules but in general there is no set of processes to follow.

Rather, one must rely on patterns found in certain integrands for clues as to how to process and maybe solve an integral.

Chapter 7 deals with some of the more readily available techniques.

We showcase 2 of them here:

(I) The Substitution Method (Anti Chain Rule)

Let $h(x) = e^{x^2+1}$. Here, $h(x)$ is a composition of differentiable functions, and

$$h'(x) = e^{x^2+1} \cdot (2x) = 2x e^{x^2+1}$$

by the Chain Rule.

But since we know that the integral of a derivative is almost the original function (up to a constant), we have

$$\int h'(x)\,dx = h(x) + C, \text{ and}$$

$$\int 2xe^{x^2+1}\,dx = e^{x^2+1} + C.$$

But in general, how does one "see" the antiderivative of a function like $2xe^{x^2+1}$?

Think of it this way:

Write $h(x) = e^{x^2+1} = F(g(x))$,

where $F(x) = e^x$ and $g(x) = x^2+1$

(we use capital F here for a reason which will be clear soon).

Then $$h'(x) = \frac{d}{dx}\left[F(g(x))\right] = F'(g(x)) \cdot g'(x)$$
$$= f(g(x)) \cdot g'(x)$$

where $f(x)$ is the derivative of $F(x)$, or $F(x)$ is the antiderivative of $f(x)$.

Now if 2 functions are equal, then their derivatives also are equal.

And if 2 functions are equal, then their antiderivatives are equal, up to a constant!

~~So~~ So if we integrate $h'(x)$, we get

$$\int h'(x)\,dx = \int \frac{d}{dx}\left[F(g(x))\right]dx = \int F'(g(x))\cdot g'(x)\,dx$$

$$\int \frac{d}{dx}\left[F(g(x))\right]dx = F(g(x)) + C = \int f(g(x))\,g'(x)\,dx$$

And this becomes the pattern:

$$\underbrace{\underbrace{\int f(g(x))}_{\text{composition of functions}}\underbrace{g'(x)}_{\text{derivative of inside function}}}_{\text{product of functions}}\,dx = \underbrace{F(g(x))}_{\text{antiderivative of outside function.}} + c$$

How one recognizes this pattern is by untangling the composition part: Create a new variable equal to the inside function:

Given $\int f(g(x)) \cdot g'(x)\,dx$, let $u = g(x)$.

Then $\frac{du}{dx} = g'(x)$. Here we are rewriting the integral in terms of a new variable u. To do this, we need to understand how u varies as we vary x. We can use $\frac{du}{dx} = g'(x)$ to write $du = g'(x)\,dx$.

(Note: $\frac{du}{dx}$ is not really a fraction here, but does look and feel like one).

Then
$$\int f(\underbrace{g(x)}_{u})\,\underbrace{g'(x)\,dx}_{du} \xlongequal[du = g'(x)dx]{u = g(x)} \underbrace{\int f(u)\,du}_{\text{easier to integrate}}$$
$$= F(u) + C$$
$$= F(g(x)) + C$$

once we return to the x-variable.

This is called the Substitution Method and helps to untangle a composition of functions in an integrand.

I call it the Anti-Chain Rule.

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ex. Find $\int 2x e^{x^2+1}\, dx$.

Here, let $u = x^2+1$, so that $du = 2x\, dx$

Then $\int \underbrace{2x\, e^{\overbrace{x^2+1}^{u}}\, dx}_{du} \overset{\substack{u = x^2+1 \\ du = 2x\,dx}}{=\!=\!=} \int e^u\, du = e^u + C$

$= e^{x^2+1} + C.$

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ex. Find the antiderivative of $6x\cos(3x^2+1)$.

Here we seek to evaluate $\int 6x\cos(3x^2+1)\,dx$, or

$\int \cos(3x^2+1)\,6x\,dx$. Try the substitution $u=3x^2+1$, $du=6x\,dx$

(the inside function of the composition in the integrand).

Then $\int \cos(\underbrace{3x^2+1}_{u})\,\underbrace{6x\,dx}_{du} = \int \cos(u)\,du = \sin u + C = \sin(3x^2+1)+C$.

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ex. Find $\int_0^{\pi/2} 3\sin^2 x\cos x\,dx$.

Here, let $u=\sin x$, $du=\cos x\,dx$. Then

$$\int 3\sin^2 x\cos x\,dx = \int 3(\underbrace{\sin x}_{u})^2\,\underbrace{\cos x\,dx}_{du} = \int 3u^2\,du$$

$$= u^3 + C = \sin^3 x + C$$

Thus the antiderivative is $\sin^3 x + C$. Hence

$$\int_0^{\pi/2} 3\sin^2 x\cos x\,dx = \sin^3 x\Big|_0^{\pi/2} = \sin^3\tfrac{\pi}{2} - \sin^3 0 = 1-0=1.$$