Class 35: Section 7.2.

In the Substitution Method,

$$\int f(g(x))\,g'(x)\,dx \;\underset{du = g'(x)dx}{\overset{u = g(x)}{=\!=}}\; \int f(u)\,du$$

$$= F(u) + C$$

$$= F(g(x)) + C$$

where $F(x)$ is the antiderivative of $f(x)$.

In the last example, we calculated the quantity $\int_0^{\frac{\pi}{2}} 3\sin^2 x \cos x \, dx$ by first calculating the antiderivative of $3\sin^2 x \cos x$, which is $\sin^3 x + C$.

Then $\int_0^{\frac{\pi}{2}} 3\sin^2 x \cos x \, dx = (\sin^3 x)\Big|_0^{\frac{\pi}{2}} = \sin^3(\frac{\pi}{2}) - \sin^3(0)$

$= (1)^3 - (0)^3 = 1.$ ∎

Notes ① One could also change the limits of integration according to the u-substitution. Then there is no need to "go back to" x.

ex In the last problem, $\int_0^{\pi/2} 3\sin^2 x\cos x\,dx$, let

$u = \sin x$, $du = \cos x\,dx$, and when $x=0$, $u=\sin 0 = 0$
$x=\frac{\pi}{2}$, $u = \sin\frac{\pi}{2} = 1$.

$$\int_0^{\frac{\pi}{2}} 3\sin^2 x\cos x\,dx = \int_0^1 3u^2\,du = u^3\Big|_0^1 = 1^3 - 0^3 = 1.$$

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② Constants can be factored in to make the substitution easier:

$$\text{ex. } \int x e^{x^2}\,dx \ \xrightarrow[\frac{1}{2}du = x\,dx]{\substack{u=x^2 \\ du = 2x\,dx \\ \text{or}}} \ \int \tfrac{1}{2}e^u\,du = \tfrac{1}{2}\int e^u\,du = \tfrac{1}{2}e^u + C = \tfrac{1}{2}e^{x^2} + C$$

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③ The substitution can also help to clean up any leftover parts of the integrand:

ex. Find $\int_0^{\sqrt{3}} x^3\sqrt{x^2+1}\,dx$.

Here, the composition $\sqrt{x^2+1}$ indicates the u-substitution is $u = x^2+1$, $du = 2x\,dx$, and when $x=0$, $u = 0^2+1 = 1$, and $x = \sqrt{3}$, $u = (\sqrt{3})^2+1 = 4$.

A straightforward substitution yields the partial result:

$$\int_0^{\sqrt{3}} x^3\sqrt{x^2+1}\,dx = \int_0^{\sqrt{3}} x^2\sqrt{\underbrace{x^2+1}_{u}}\;\underbrace{x\,dx}_{du} = \int_1^4 \frac{x^2\sqrt{u}}{2}\,du$$

We are not done, as we need to render the remaining x^2 part as a function of u:

Here since $u = x^2+1$, $x^2 = u-1$, and

$$\frac{1}{2}\int_1^4 x^2\sqrt{u}\,du = \frac{1}{2}\int_1^4 (u-1)\sqrt{u}\,du = \frac{1}{2}\int_1^4 (u^{3/2} - u^{1/2})\,du$$

$$= \frac{1}{2}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right)\Big|_1^4 = \frac{1}{2}\left(\frac{2}{5}(4)^{5/2} - \frac{2}{3}(4)^{3/2}\right) - \frac{1}{2}\left(\frac{2}{5}(1)^{5/2} - \frac{2}{3}(1)^{3/2}\right)$$

$$= \frac{1}{2}\left(\frac{2}{5}(32) - \frac{2}{3}(8)\right) - \frac{1}{2}\left(\frac{2}{5} - \frac{2}{3}\right) = \cdots$$ ∎

Yet another Anti-Diff Rule.

Recall the Product Rule for Differentiation.

$$\frac{d}{dx}[f(x)\,g(x)] = f'(x)\,g(x) + f(x)\,g'(x).$$

Rewrite this as

$$(\star) \quad f(x)\,g'(x) = \frac{d}{dx}[f(x)\,g(x)] - f'(x)\,g(x)$$

Note: ⓐ If 2 functions are equal, so are their derivatives.

ⓑ If 2 fncs are equal, so are their antiderivatives, only up to a constant !!!

Hence we can integrate the above equation $(\star)$ to get

$$\int \underbrace{f(x)}_{u}\,\underbrace{g'(x)\,dx}_{dv} = \underbrace{\int \frac{d}{dx}[\underbrace{f(x)}_{u}\,\underbrace{g(x)}_{v}]\,dx}_{uv} - \int \underbrace{g(x)}_{v}\,\underbrace{f'(x)\,dx}_{du}.$$

Or, in simpler terms, given the double substitution

$$u = f(x), \quad du = f'(x)\,dx,$$

$$v = g(x), \quad dv = g'(x)\,dx,$$

we get $\boxed{\int u\,dv = uv - \int v\,du}$,

a shorthand way to view this equation.

This is a new possible way to evaluate an integral that is a product of functions; by replacing it with a different, hopefully easier to handle, integral.

This replacement is called Integration by Parts

It comes from the Product Rule. I call it the Anti-Product Rule.

ex. Find the anti derivative of xe^{2x}.

Here, $\int xe^{2x}\,dx$ is not obvious and a u-substitution will not work. Try it!

Via IbP, try the double substitution
$$u = x \quad \text{and} \quad dv = e^{2x}\,dx.$$

Fill in the missing data:
$$du = dx \quad \text{and} \quad v = \tfrac{1}{2}e^{2x}$$

Then
$$\int xe^{2x}\,dx = \int u\,dv = uv - \int v\,du$$
$$= \tfrac{1}{2}xe^{2x} - \int \tfrac{1}{2}e^{2x}\,dx$$
$$= \tfrac{1}{2}xe^{2x} - \tfrac{1}{4}e^{2x} + C.$$

Is it correct? Take the derivative to see.
$$\frac{d}{dx}\left[\tfrac{1}{2}xe^{2x} - \tfrac{1}{4}e^{2x} + C\right] = \tfrac{1}{2}e^{2x} + \tfrac{1}{2}x(2e^{2x}) - \tfrac{1}{4}(2e^{2x}) = xe^{2x}.$$

☺

ex. Find $\int x\cos x\,dx$. Here try

$u = x$, $dv = \cos x\,dx$. Then

$du = dx$, $v = \sin x$, and

$$\int \underbrace{x}_{u}\,\underbrace{\cos x\,dx}_{dv} = x\sin x - \int \sin x\,dx$$

$$= x\sin x + \cos x + C.$$

Does this work? Check!

Notes ① Sometimes, IbP is needed repeatedly:

ex. $$\int x^2 \sin x\,dx \overset{\substack{u=x^2\\ dv=\sin x\,dx}}{\underset{\substack{du=2x\,dx\\ v=-\cos x}}{=\!=\!=}} x^2(-\cos x) - \int(-\cos x)\,2x\,dx$$

$$= -x^2\cos x + 2\int x\cos x\,dx.$$

(like the previous problem, now do IbP again on last term)

② With practice, patterns emerge as to which substitution to make: For example, if one factor in the integrand is a polynomial, try making that u. Then the du on the right will be a polynomial of less degree.

<u>But just try stuff!</u>

③ Sometimes one of the substitutions for u or v is simply the function 1.

ex. $\int \ln x \, dx$. (What is the antiderivative of $\ln x$?)

We solve this by IBP: Let $u = \ln x$, $dv = dx$ so $du = \frac{1}{x} dx$ and $v = x$. Then

$$\int \underbrace{\ln x}_{u} \underbrace{dx}_{dv} = \underbrace{x}_{v} \underbrace{\ln x}_{u} - \int \underbrace{x}_{v} \underbrace{\left(\tfrac{1}{x}\right) dx}_{du} = x \ln x - \int dx$$

$$= x \ln x - x + C.$$

Does it work?

(4) <u>Last example</u>: Sometimes one can go in circles by IBP. But we can still solve algebraically:

ex. Find $\int e^x \sin x \, dx$.

Here, try IBP with $u = e^x$, $dv = \sin x \, dx$, so $du = e^x dx$ and $v = -\cos x$. We get

$$\int e^x \sin x \, dx = -e^x \cos x + \underbrace{\int e^x \cos x \, dx}_{\text{IBP with } u = e^x,\ dv = \cos x dx,\ du = e^x dx,\ v = \sin x.}$$

$$= -e^x \cos x + \left(e^x \sin x - \underbrace{\int e^x \sin x \, dx}_{\text{back where we started? Not really! Solve for this unknown}}\right)$$

bring to other side

$$2\int e^x \sin x \, dx = e^x \sin x - e^x \cos x, \text{ or}$$

$$\int e^x \sin x \, dx = \frac{1}{2}\left(e^x \sin x - e^x \cos x\right) + C.$$

Check by differentiating whether this is the antiderivative.

∎

(5) For definite integrals, all limits apply at the time they are needed:

$$\int_a^b \underbrace{f(x)}_{u}\, \underbrace{g'(x)\,dx}_{dv} = \Big[\underbrace{f(x)}_{u}\, \underbrace{g(x)}_{v}\Big]\Big|_a^b - \int_a^b \underbrace{g(x)}_{v}\, \underbrace{f'(x)\,dx}_{du}.$$