EXAMPLE: USING ABEL'S THEMREM TO HELP SOLVE A SECOND-ORDER, LINEAR HOMOGENEOUS ODE

110.302 DIFFERENTIAL EQUATIONS
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Given a second order, linear, homogeneous differential equation

$$y'' + p(t)y' + q(t)y = 0,$$

where both $p(t)$ and $q(t)$ are continuous on some open t-interval I, and two solutions $y_1(t)$ and $y_2(t)$, one can form a fundamental set of solutions as the linear combination of these two

$$y(t) = c_1y_1(t) + c_2y_2(t)$$

ONLY under the condition that the Wronskian determinant

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} \neq 0$$

for all $t \in I$. This condition implies that the two differentiable functions $y_1(t)$ and $y_2(t)$ are independent. But regardless of independence of any two solutions to a second-order, linear, homogeneous ODE, we have Abel's Theorem, which essentially says that the Wronskian determinant always has a certain form:

Theorem (Abel's Theorem)**.** *If $y_1(t)$ and $y_2(t)$ are two solutions to the ODE $y'' + p(t)y' + q(t)y = 0$, where $p(t)$ and $q(t)$ are continuous on some open t-interval I, then*

$$W(y_1, y_2)(t) = Ce^{-\int p(t)\, dt}$$

where C depends on the choice of y_1 and y_2, but not on t.

Example 1. Solve the ODE $y'' + 2y' + 1 = 0$.

Strategy. Use the characteristic equation to find the general solution.

Solution. Here, the characteristic equation is $r^2 + 2r + 1 = 0$, which can be rewritten as $(r + 1)^2 = 0$, and is solved by $r_1 = -1 = r_2$. An attempt at a general solution would immediately yield

$$y(t) = c_1e^{-t} + c_2e^{-t}.$$

However, this does not work since the two solutions $y_1(t) = e^{-t}$ and $y_2(t) = e^{-1t}$ have as their Wronskian

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{-t} & e^{-t} \\ -e^{-t} & -e^{-t} \end{vmatrix} = -e^{-2t} + e^{-2t} = 0, \quad \text{for all } t \in \mathbb{R}.$$

So simply trying to use the characteristic equation directly will not work. So let's try a different strategy.

Strategy. Use Abels' Theorem to construct the second independent solution to the ODE.

Solution. We know from Abel's Theorem that, for any two solutions $y_1(t)$ and $y_2(t)$ to the ODE $y'' + p(t)y' + q(t)y = 0$, we have

$$W(y_1, y_2)(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = Ce^{-\int p(t)\,dt}.$$

In our case, we have that $p(t) = 2$, and $y_1(t) = e^{-t}$. This means that

$$e^{-\int p(t)\,dt} = e^{-\int 2\,dt} = e^{-2t}.$$

We also know that, the value of the constant C will depend only on y_1 and y_2. So let's attempt to find the value of the Wronskian that corresponds to $C = 1$. So set $C = 1$ and $y_1(t) = e^{-t}$. We get

$$\begin{aligned} y_1(t)y_2'(t) - y_1'(t)y_2(t) &= Ce^{-\int p(t)\,dt} \\ e^{-t}y_2'(t) - (-e^{-t})y_2(t) &= 1 \cdot e^{-2t}, \end{aligned}$$

which simplifies to

$$y_2' + y_2 = e^{-t}.$$

This is a first-order ODE in $y_2(t)$, and is another kind of Reduction of Order technique that is useful here.

We can solve this as a first-order linear ODE, using an integrating factor, which in this case is $e^{\int 1\,dt} = e^t$, so that

$$\begin{aligned} e^t\left[y_2' + y_2 \right. &= \left. e^{-t}\right] \\ e^t y_2' + e^t y_2 &= 1 \\ \frac{d}{dt}\left[e^t y_2\right] &= 1, \end{aligned}$$

which we can integrate, with respect to t, to get

$$e^t y_2 = t + K.$$

Solving for y_2, we get

$$y_2(t) = te^{-t} + Ke^{-t}.$$

Here is a question: Does y_2 solve the original ODE $y'' + 2y' + y = 0$? The answer here is yes, since

$$\begin{aligned} y_2'' + 2y_2' + y_2 &= (K-2)e^{-t} + te^{-t} + 2\left((1-K)e^{-t} - te^{-t}\right) + te^{-t} + Ke^{-t} \\ &= Ke^{-t} - 2e^{-t} + te^{-t} + 2e^{-t} - 2Ke^{-t} - 2te^{-t} + te^{-t} + Ke^{-t} \\ &= 0. \end{aligned}$$

So we can construct our general solution as

$$\begin{aligned} y(t) &= c_1y_1(t) + c_2y_2(t) = c_1e^{-t} + c_2\left(te^{-t} + Ke^{-t}\right) \\ &= C_1e^{-t} + C_2te^{-t}, \end{aligned}$$

since we can combine constants so that $C_1 = c_1 + K$, and $C_2 = c_2$. Note that we can just take $y_1(t) = e^{-t}$ and $y_2(t) = te^{-t}$ to form our general solution here.

The next question is: Are these two solutions $y_1(t) = e^{-t}$ and $y_2(t) = te^{-t}$ independent, so that the form of the general solution is justified? We check:

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & e^{-t} - te^{-t} \end{vmatrix} = e^{-2t} - te^{-2t} + te^{-2t} = e^{-2t} \neq 0$$

on all of $\mathbb{R}$. Hence these two functions, as solutions to the original ODE, are independent, and the general solution is the linear combination of them.