EXAMPLE: EXACT DIFFERENTIAL EQUATIONS

110.302 DIFFERENTIAL EQUATIONS
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Problem. Solve the Ordinary Differential Equation $\frac{dy}{dx} = \frac{x^2}{1-y^2}$.

Strategy. Solving the ODE means finding the general solution (the 1-parameter family of solutions). We first note that it is a separable differential equation. But also, it is exact. We will solve this problem both ways.

Solution. This ODE is separable since the right-hand-side can be written as a product of two functions, one solely a function of the independent variable x and the other of the dependent variable y. Here, we can write

$$\frac{dy}{dx} = \frac{x^2}{1-y^2} = x^2\left(\frac{1}{1-y^2}\right).$$

We can separate the variables by dividing the entire equation by the function of the dependent variable:

$$(1-y^2)\left[\frac{dy}{dx} = x^2\left(\frac{1}{1-y^2}\right)\right]$$
$$(1-y^2)\frac{dy}{dx} = x^2.$$

Now we can integrate both sides with respect to x

$$\int (1-y^2)\frac{dy}{dx}\,dx = \int (1-y^2)\,dy = \int x^2\,dx$$
$$y - \frac{y^3}{3} = \frac{x^3}{3} + C.$$

This is the implicit solution to the ODE.

This ODE is also exact. To see this, rewrite the equation in the general form $M(x,y) + N(x,y)\frac{dy}{dx} = 0$. Here,

$$-x^2 + (1-y^2)\frac{dy}{dx} = 0.$$

Recall in the book that a separable ODE is one in the general form where M is solely a function of x and N is solely a function of y. You can see that this is the case, and the ODE is separable.

The criterion for the equation $M(x,y) + N(x,y)\frac{dy}{dx} = 0$ to be exact is for the partial of $M(x,y)$ with respect to y to be equal to the partial of $N(x,y)$ with respect to x, or

$$\frac{\partial M}{\partial y} = M_y = N_x = \frac{\partial N}{\partial x}.$$

However, whenever a ODE is separable, there is no y in the function M and there is no x in the function N: An ODE is separable if it can be written

$$M(x) + N(y)\frac{dy}{dx} = 0.$$

In our case, $M(x,y) = M(x) = -x^2$, and $N(x,y) = N(y) = 1 - y^2$. Thus

$$M_y = 0 = N_x$$

and the ODE is exact.

Note. *Separable first-order ODEs are ALWAYS exact. But many exact ODEs are NOT separable.*

Thus there exists a function $\varphi(x,y)$ which solves the ODE implicitly, and whose partials are the functions M and N. To solve, identify the partial of φ with respect to x with M and integrate with respect to x (to recover φ):

$$\frac{\partial \varphi}{\partial x} = -x^2, \qquad \text{and} \quad \varphi(x,y) = \int \frac{\partial \varphi}{\partial x}\,dx = \int (-x^2)\,dx = -\frac{x^3}{3} + h(y).$$

So we now have at least some information about the form of the function $\varphi(x,y)$.

Question 1. *Why is the constant of integration here the function $h(y)$? This is a very important question!*

Now if we take our form for $\varphi(x,y) = -\frac{x^3}{3} + h(y)$, and take the partial with respect to y, we get

$$\frac{\partial \varphi}{\partial y}(x,y) = \frac{\partial}{\partial y}\left[-\frac{x^3}{3} + h(y)\right] = 0 + h'(y).$$

But the partial of φ with respect to y is also precisely the function $N(y) = 1 - y^2$. Hence we equate the two

$$h'(y) = 1 - y^2.$$

Thus using Calculus II, we can find the form for $h(y)$: We get $h(y) = y - \frac{y^3}{3}$, so that

$$\varphi(x,y) = -\frac{x^3}{3} + y - \frac{y^3}{3}.$$

Finally, the entire original ODE $M(x,y) + N(x,y)\frac{dy}{dx} = 0$ is simply a restatement that the total derivative with respect to the independent variable x, assuming y is an implicit function of x, is zero. This happens along the level curves of $\varphi(x,y)$:

$$\frac{d}{dx}\varphi(x,y(x)) = 0 = \frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial y}\frac{dy}{dx} = -x^2 + (1-y^2)\frac{dy}{dx}.$$

Thus the general solution to the original ODE is

$$\varphi(x,y) = C = -\frac{x^3}{3} + y - \frac{y^3}{3},$$

as before.