Problem. Solve the Ordinary Differential Equation \( \frac{dy}{dx} = \frac{x^2}{1 - y^2} \).

Strategy. Solving the ODE means finding the general solution (the 1-parameter family of solutions). We first note that it is a separable differential equation. But also, it is exact. We will solve this problem both ways.

Solution. This ODE is separable since the right-hand-side can be written as a product of two functions, one solely a function of the independent variable \( x \) and the other of the dependent variable \( y \). Here, we can write

\[
\frac{dy}{dx} = \frac{x^2}{1 - y^2} = x^2 \left( \frac{1}{1 - y^2} \right).
\]

We can separate the variables by dividing the entire equation by the function of the dependent variable:

\[
(1 - y^2) \left[ \frac{dy}{dx} = x^2 \left( \frac{1}{1 - y^2} \right) \right]
\]

\[
(1 - y^2) \frac{dy}{dx} = x^2.
\]

Now we can integrate both sides with respect to \( x \)

\[
\int (1 - y^2) \frac{dy}{dx} \, dx = \int (1 - y^2) \, dy = \int x^2 \, dx
\]

\[
y - \frac{y^3}{3} = \frac{x^3}{3} + C.
\]

This is the implicit solution to the ODE.

This ODE is also exact. To see this, rewrite the equation in the general form \( M(x, y) + N(x, y) \frac{dy}{dx} = 0 \). Here,

\[
-x^2 + (1 - y^2) \frac{dy}{dx} = 0.
\]
Recall in the book that a separable ODE is one in the general form where $M$ is solely a function of $x$ and $N$ is solely a function of $y$. You can see that this is the case, and the ODE is separable.

The criterion for the equation $M(x, y) + N(x, y)\frac{dy}{dx} = 0$ to be exact is for the partial of $M(x, y)$ with respect to $y$ to be equal to the partial of $N(x, y)$ with respect to $x$, or

$$\frac{\partial M}{\partial y} = M_y = N_x = \frac{\partial N}{\partial x}.$$ 

However, whenever a ODE is separable, there is no $y$ in the function $M$ and there is no $x$ in the function $N$: An ODE is separable if it can be written

$$M(x) + N(y)\frac{dy}{dx} = 0.$$ 

In our case, $M(x, y) = M(x) = -x^2$, and $N(x, y) = N(y) = 1 - y^2$. Thus

$$M_y = 0 = N_x$$

and the ODE is exact.

**Note.** *Separable first-order ODEs are ALWAYS exact. But many exact ODEs are NOT separable.*

Thus there exists a function $\varphi(x, y)$ which solves the ODE implicitly, and whose partials are the functions $M$ and $N$. To solve, identify the partial of $\varphi$ with respect to $x$ with $M$ and integrate with respect to $x$ (to recover $\varphi$):

$$\frac{\partial \varphi}{\partial x} = -x^2,$$

and

$$\varphi(x, y) = \int \frac{\partial \varphi}{\partial x} \, dx = \int (-x^2) \, dx = -\frac{x^3}{3} + h(y).$$

So we now have at least some information about the form of the function $\varphi(x, y)$.

**Question 1.** Why is the constant of integration here the function $h(y)$? This is a very important question!

Now if we take our form for $\varphi(x, y) = -\frac{x^3}{3} + h(y)$, and take the partial with respect to $y$, we get

$$\frac{\partial \varphi}{\partial y}(x, y) = \frac{\partial}{\partial y} \left[ -\frac{x^3}{3} + h(y) \right] = 0 + h'(y).$$

But the partial of $\varphi$ with respect to $y$ is also precisely the function $N(y) = 1 - y^2$. Hence we equate the two

$$h'(y) = 1 - y^2.$$
Thus using Calculus II, we can find the form for \( h(y) \): We get \( h(y) = y - \frac{y^3}{3} \), so that
\[
\varphi(x, y) = -\frac{x^3}{3} + y - \frac{y^3}{3}.
\]

Finally, the entire original ODE \( M(x, y) + N(x, y)\frac{dy}{dx} = 0 \) is simply a restatement that the total derivative with respect to the independent variable \( x \), assuming \( y \) is an implicit function of \( x \), is zero. This happens along the level curves of \( \varphi(x, y) \):
\[
\frac{d}{dx}\varphi(x, y(x)) = 0 = \frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y} \frac{dy}{dx} = -x^2 + (1 - y^2) \frac{dy}{dx}.
\]

Thus the general solution to the original ODE is
\[
\varphi(x, y) = C = -\frac{x^3}{3} + y - \frac{y^3}{3},
\]
as before.