110.302 ORDINARY DIFFERENTIAL EQUATIONS

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Existence and Uniqueness worksheet

Consider the first order IVP

$$\dot{y}(t) = f(t, y), \quad y(t_0) = y_0. \tag{1}$$

As talked about in class, the question of whether Equation 1 has a solution, and when it has a solution, if it is uniquely defined, is a difficult one in general. However, due to the following theorem, the properties of $f(t, y)$ at and near the initial point (t_0, y_0) can ensure that unique solutions exist:

Theorem 1. *Suppose $f(t, y)$ and $\frac{\partial f}{\partial y}(t, y)$ are continuous in some rectangle*

$$R = \left\{ (t, y) \in \mathbb{R}^2 \mid \alpha < t < \beta, \gamma < y < \delta \right\},$$

containing the initial point (t_0, y_0). Then, in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \phi(t)$ of Equation 1.

To give a good sense of why this is true, let's start with a definition:

Definition 2. An *operator* is a function whose domain and range are functions.

A good example of this is the derivative operator $\frac{d}{dx}$ which acts on all differentiable functions of one independent variable, and takes them to other (in this case, at least) continuous functions. Think

$$\frac{d}{dx}(x^2 + \sin x) = 2x + \cos x.$$

There are numerous technical difficulties in defining operators correctly, but for now, simply accept this general description.

We claim that any possible solution $y = \phi(t)$ (if it exists) to Equation 1 must satisfy

$$\phi(t) = y_0 + \int_{t_0}^{t} f(s, \phi(s))\, ds \tag{2}$$

for all t in some interval containing t_0.

Exercise 1. Show that this is true (really, this is very straightforward. Simply take the derivative of Equation 2, noting that the right-hand side is easy to derive knowing the Fundamental Theorem of Calculus.)

At this point, existence of a solution to the ODE is assured in the case that $f(t, y)$ is continuous on R, as the integral in Equation 2 will then exist at least on some smaller interval $t_0 - h < t < t_0 + h$ contained inside $\alpha < t < \beta$ (the reason it may not exist all the way out to the edge of R? What if the edge of R is an asymptote in the t variable?)

As for uniqueness, suppose $f(t, y)$ is continuous as above, and consider the following operator T, which takes a function $\phi(t)$ to its image $T(\phi(t))$ defined by

$$T(\phi(t)) = y_0 + \int_{t_0}^{t} f(s, \phi(s))\, ds.$$

We can stick in many functions for $\phi(t)$ and the image will be a different function $T(\phi(t))$ (sometimes, we will simply write $T(\phi)$) which is still a function of t. However, looking back at Equation 2, if we stick in the function $\phi(t)$ which solves our IVP, the image $T(\phi)$ should be the same as ϕ. In this case, we call such a function a *fixed point* of T, since $T(\phi) = \phi$.

Example 3. Consider the Initial Value Problem $y' = ty$, $y(0) = 1$. This ODE is separable, and you should verify that the particular solution in $y(t) = e^{t^2/2}$. According to the Existence and Uniqueness Theorem, this will be the ONLY solution passing through the point $(t_0, y_0) = (0, 1)$ in the ty-plane.

If we define the operator T as above, then for THIS ODE, we get $f(s, \phi(s)) = s\phi(s)$, $t_0 = 0$, and $y_0 = 1$, and

$$T(\phi) = y_0 + \int_{t_0}^{t} f(s, \phi(s))\ ds = 1 + \int_{0}^{t} s\phi(s)\ ds.$$

Let's input a few functions into this operator and "see" where they go:

- Let $\phi(t) = 2$, a constant: Then $T(\phi) = T\,[2]$, and

$$T\,(2) = 1 + \int_0^t 2s\ ds = 1 + s^2\Big|_0^t = 1 + t^2.$$

- Let $\phi(t) = t^2$: Then

$$T(\phi) = T\left(t^2\right) = 1 + \int_0^t s(s^2)\ ds = 1 + \int_0^t s^3)\ ds = 1 + \frac{s^4}{4}\Big|_0^t = 1 + \frac{t^4}{4}.$$

- Let $\phi(t) = \cos t$: Then

$$\begin{aligned} T(\phi) = T\,(\cos t) &= 1 + \int_0^t s\ \cos s\ ds \\ &= 1 + s\ \sin s\Big|_0^t - \int_0^t \sin s\ ds \\ &= 1 + t\sin t + \cos t - 1 = t\sin t + \cos t. \end{aligned}$$

- Let $\phi(t) = e^t$: Then

$$\begin{aligned} T(\phi) = T\left(e^t\right) &= 1 + \int_0^t se^s\ ds \\ &= 1 + se^s\Big|_0^t - \int_0^t e^s\ ds \\ &= 1 + te^t - e^t + 1 = 2 - e^t + te^t. \end{aligned}$$

- Let $\phi(t) = e^{t^2/2}$: Then

$$T(\phi) = T\left(e^{t^2/2}\right) = 1 + \int_0^t s e^{s^2/2}\, ds$$
$$= 1 + e^{s^2/2}\Big|_0^t = 1 + e^{t^2/2} - 1 = e^{t^2/2}.$$

This last input function seems to be the only one where $T(\phi(t)) = \phi(t)$. That is, it is the only example here of a fixed point for this operator.

Exercise 2. Find ALL fixed points for the derivative operator $\dfrac{d}{dx}$ on the domain $\mathbb{R}$.

Hence, instead of looking for solutions to the IVP, we can instead look for fixed points of the operator T, since any fixed point for T will also satisfy Equation 2 and hence solve the IVP. How do we do this? Fortunately, this operator has an interesting property. First, for T an operator and ϕ a function, define

$$T^n(\phi) = \overbrace{T(T(\cdots(T(\phi))\cdots))}^{n \text{ times}}.$$

Incidentally, this is called iterating the function T, and the above expression is called the nth iterate of ϕ under T.

Theorem 4. *Suppose you have a way to measure the distance between two functions $f(t)$ and $g(t)$ and call this distance $dist(f, g)$. If an operator T satisfies*

$$dist\,(T(f), T(g)) \leq C \cdot dist(f, g), \quad \text{for some } 0 < C < 1,$$

then there is a single function ϕ that satisfies $T(\phi) = \phi$. In addition, this unique fixed point satisfies

$$\phi = \lim_{n\to\infty} T^n(g)$$

for any starting function $g(t)$.

Remark 5. Any operator that satisfies the distance criterion in this theorem is called a C-contraction, and in essence this theorem is called the Contraction Principle, or the Contraction Mapping Theorem; a common tool used in the study of ODEs and Dynamical Systems.

Remark 6. Though not entirely necessary, it does make the proof easier to suppose that both $f(t, y)$ and $\frac{\partial f}{\partial y}(t, y)$ are not only continuous on R, but bounded here also. This is because we can always slightly restrict R at an edge where one of the variables blows up. The proof is true even in this case. However, it is much easier to see with this restriction. As an example, let $f(t, y) = \log y$. Here, both f and $\frac{\partial f}{\partial y} = \frac{1}{y}$ are continuous on the rectangle $-1 < t < 1$, $0 < y < 1$. However, neither are bounded here. Create a new rectangle $\widetilde{R}$ by moving the left boundary of R slightly to the right; for a small $\epsilon > 0$, define $\widetilde{R}$ to be $-1 < t < 1$, $\epsilon < y < 1$. Here then both f and $\frac{\partial f}{\partial y}$ are continuous and bounded on $\widetilde{R}$.

proof of Theorem 1. Under the supposition that f and $\frac{\partial f}{\partial y}$ are bounded on R, call

$$M = \max_R \left| \frac{\partial f}{\partial y}(t, y) \right|,$$

and choose a small number $h = \frac{C}{M}$, where $C < 1$. Then define a distance within the set of continuous functions on the closed interval $I = [t_0 - h, t_0 + h]$ by

$$dist(g, h) = \max_{t \in I} \left| g(t) - h(t) \right|.$$

Then we have

$$
\begin{aligned}
(3) \qquad dist\,(T(g), T(h)) &= \max_{t\in I} \left| T(g(t)) - T(h(t)) \right| \\
(4) \qquad &= \max_{t\in I} \left| y_0 + \int_{t_0}^{t} f(s, g(s))\, ds - y_0 - \int_{t_0}^{t} f(s, h(s))\, ds \right| \\
(5) \qquad &= \max_{t\in I} \left| \int_{t_0}^{t} f(s, g(s)) - f(s, h(s))\, ds \right| \\
(6) \qquad &= \max_{t\in I} \left| \int_{t_0}^{t} \left[\int_{h(s)}^{g(s)} \frac{\partial f}{\partial y}(s, r)\, dr \right] ds \right| \\
(7) \qquad &\le \max_{t\in I} \left| \int_{t_0}^{t} M \left| g(s) - h(s) \right| \, ds \right| \\
(8) \qquad &\le \max_{t\in I} \int_{t_0}^{t} M \cdot dist(g, h)\, ds \\
(9) \qquad &\le \max_{t\in I} \left\{ M \cdot dist(g, h) \cdot |t - t_0| \right\}
\end{aligned}
$$

Exercise 3. The justifications of going from Step 5 to Step 6 and from Step 6 to Step 7 are adaptations of major Theorems from Calculus I-II to functions of more than one independent variable. Find what theorems these are and show that these are valid justifications.

Exercise 4. Justify why the remaining steps are true.

Now notice is the last inequality that since $I = [t_0 - h, t_0 + h]$, we have that

$$|t - t_0| \le h = \frac{c}{M}.$$

Hence

$$
\begin{aligned}
dist\,(T(g), T(h)) &\le \max_{t\in I} \left\{ M \cdot dist(g, h) \cdot |t - t_0| \right\} \\
&= M \cdot dist(g, h) \cdot \frac{C}{M} = C \cdot dist(g, h).
\end{aligned}
$$

Hence T is a C-contraction and there is a unique fixed point ϕ (which is a solution to the original IVP) on the interval I. Here

$$\phi(t) = T(\phi(t)) = y_0 + \int_{t_0}^{t} f(s, \phi(s))\, ds.$$

□

As an application, we can actually use this construction to "solve" an ODE:

Example 7. Solve the IVP

$$y' = 2t(1+y), \quad y(0) = 0.$$

Here, $f(t,y) = 2t(1+y)$, as well as $\frac{\partial f}{\partial y}(t,y) = 2t$ are both continuous on the whole plane $\mathbb{R}^2$. Hence unique solutions exist everywhere.

To actually find a solution, start with an initial guess to be

$$\phi_0(t) = 0.$$

Notice that this choice of $\phi_0(t)$ does not solve the ODE. But since the operator T is a contraction, iterating will lead us to a solution: Define $T(\phi_0(t)) = \phi_1(t)$, and similarly, define

$$\phi_n(t) = T(\phi_{n-1}(t)) = \overbrace{T(T(\cdots(T(\phi_0(t)))\cdots))}^{n \text{ times}}.$$

Here

$$\phi_1(t) = T(\phi_0(t)) = y_0 + \int_0^t 2s(1+\phi_0(s))\, ds = \int_0^t 2s(1+0)\, ds = t^2.$$

Continuing, we get

$$\begin{aligned}
\phi_2(t) &= T(\phi_1(t)) = y_0 + \int_0^t 2s(1+\phi_1(s))\, ds = \int_0^t 2s(1+s^2)\, ds = t^2 + \frac{1}{2}t^4, \\
\phi_3(t) &= T(\phi_2(t)) = y_0 + \int_0^t 2s(1+\phi_2(s))\, ds = \int_0^t 2s\left(1+s^2+\frac{1}{2}s^4\right) ds = t^2 + \frac{1}{2}t^4 + \frac{1}{6}t^6, \\
\phi_4(t) &= T(\phi_3(t)) = y_0 + \int_0^t 2s(1+\phi_3(s))\, ds = \int_0^t 2s\left(1+s^2+\frac{1}{2}s^4+\frac{1}{6}t^6\right) ds = t^2 + \frac{1}{2}t^4 + \frac{1}{6}t^6 + \frac{1}{24}t^8.
\end{aligned}$$

Exercise 5. Find the pattern and write out a finite series expression for $\phi_n(t)$. Here one can prove by induction that the pattern you find is the nth iterate function. However, I am more interested in you "seeing" it right now.

Exercise 6. Find a closed form expression for $\lim_{n\to\infty} \phi_n(t)$ and show that it is a solution of the IVP.