PROBLEM SET 4, QUESTION 6: THE FISH PROBLEM

110.302 DIFFERENTIAL EQUATIONS
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Question 6. Consider the population model for a species of fish in a lake

$$\frac{dP}{dt} = 2P - \frac{P^2}{50},$$

where P is measured in thousands of fish and t is measured in years. The US Fish and Wildlife Service, which is managing the lake, wants to issue fishing licenses for the harvesting of some of the fish (this amounts to a constant term being subtracted off of the right hand side above, which is a function of h, the number of licenses issued). Each fishing license is valid for the annual take of 3000 fish. Draw a bifurcation diagram for the above ODE with the added parameter part, and answer the following questions.

(a) What is the largest number of licenses that can be issued if the goal is to keep a stable population of fish in the lake over the long term?

(b) If the largest number of licenses is actually issued, what is the expected long term stable population of fish in the lake?

(c) Solve the IVP given by the above differential equation and the initial value $P(0) = 2$ (This corresponds to an initial population of 2000 fish in the lake, and an assumption that there will be no harvesting, $h = 0$).

(d) As an expert consultant to the USFWS, discuss the ramifications of issuing the maximal number of licenses allowed by a mathematical model in the presence of real world issues which may temporary affect populations (drought, flooding, unlawful fishing, pollution, etc.)

(e) What is your final recommendation, in terms of the number of licenses that should be issued, to the USFWS? Back this final recommendation up with sound reasoning.

Solution. To answer these questions, you must understand how the harvesting affects the population. Since P is measured in thousands of fish, and each license allows for 3000 fish to be harvested, if we let h be the number of licenses issued, then the constant term subtracted off of the ODE is $3h$, and the ODE is then

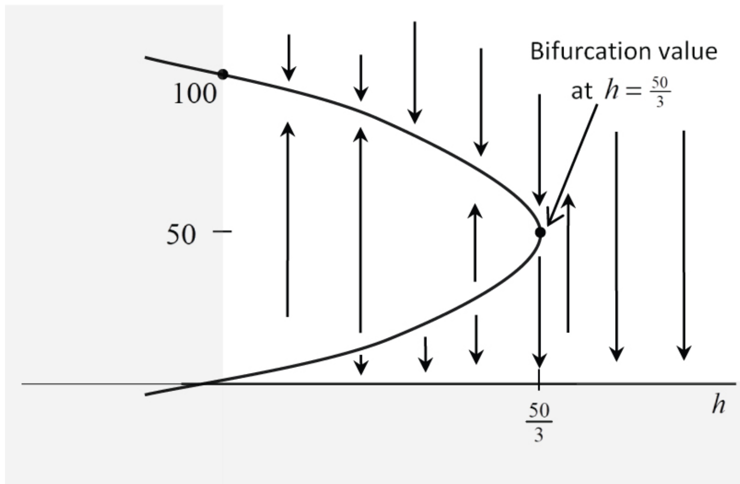
$$\frac{dP}{dt} = -\frac{P^2}{50} + 2p - 3h.$$

The variable h is a parameter and affects any equilibria of this first order, autonomous ODE. The equilibria are found by setting the right hand side to 0 and solving for P:

$$-\frac{P^2}{50} + 2p - 3h = 0 \quad \Rightarrow P = \frac{-2 \pm \sqrt{4 - \frac{12h}{50}}}{2\left(-\frac{1}{50}\right)} = 50\left(1 \pm \sqrt{1 - \frac{3h}{50}}\right).$$

It should be easy to see that whether the term under the radical is positive, zero or negative will determine respectively the number of equilibria for any given value of h; This leads to $h < \frac{50}{3}$, where there are 2 solutions, $h = \frac{50}{3}$, with 1 solution (the radical vanishes), and $h > \frac{50}{3}$, where there are no solutions.

Thus the bifurcation value for h is $h = \frac{50}{3}$. The graph of the equation $P = 50\left(1 \pm \sqrt{1 - \frac{3h}{50}}\right)$ in the hP-plane is the basic construction of the bifurcation diagram:



Things to note about this diagram:

(1) The shaded region is not relevant to the problem since negative populations or negative licenses issuances do not make sense.
(2) The stability arrows are found by setting a value for h and drawing the phase line.
(3) For each value of h between $h = 0$ and $h = \frac{50}{3}$, there is a stable equilibrium and an unstable one. As long as our initial fish population is above the unstable curve (the lower branch of the parabola), then the long term fish population will approach the upper equilibrium along the vertical line corresponding to the value of h.

To answer the questions:

(a) Since license issuances are whole numbers, the maximum number of fish licenses that can be issued for which a stable population is possible is the first whole number below the bifurcation point. Here, $h = 16$ licenses.
(b) The stable population is the larger of the two solutions to $P = 50\left(1 \pm \sqrt{1 - \frac{3h}{50}}\right)$ when $h = 16$. Thus, $P = 50\left(1 + \sqrt{.04}\right) = 50(1.2) = 60$, or 60000 fish.
(c) I will leave the actual ODE solving to you. But note that the actual solution to the ODE is not needed for a proper analysis and conclusion to this problem.
(d) This answer is up to you. However, some ramifications to consider. If the USFWS issues 16 licenses (the maximum), then if the environment changes in the lake due to drought or pollution, the population may not be able to recover easily. This could be modeled as an additional term subtracted off of the ODE, and may lower the bifurcation value. You could lose the entire population. Another thing to consider is poaching (fishing without a license). If something like 9000 fish are poached each year, then this would be equivalent to issuing an additional 3 licenses. That would look like a total of 19 licenses issued

which would be far too much to sustain any population. And a third consideration is that, if we start with 2000 fish, then issuing 16 licenses in the first year will kill off the population, right? One would have to cycle up the number of licenses. Maybe offer none for a few years to stabilize the population. Then start selling licenses. There are many other possibilities in this category.

(e) You are the consultant. What do you say?