WHY A LINEAR CENTER DOES NOT INDICATE AN NONLINEAR ONE.

110.302 DIFFERENTIAL EQUATIONS
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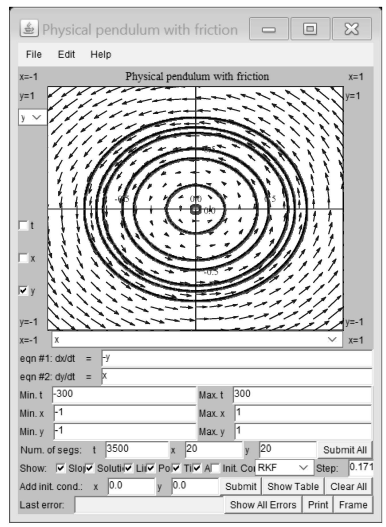
Here is an example of the failure of a linearized center to indicate the type and/or stability of a nonlinear equilibrium.

Let $\dot{x} = -y$ and $\dot{y} = x$. Then this linear system is

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x},$$

and the characteristic equation $r^2 + 1 = 0$ yields the solutions $r = \pm i$, resulting in a center at the origin.



Now, though, consider the modified (and nonlinear) ODE system

$$\begin{aligned} \dot{x} &= -y - x^3 &= F(x, y) \\ \dot{y} &= x - y^3 &= G(x, y). \end{aligned}$$

Here, the system is certainly almost linear (the two functions F and G are polynomials in x and y) everywhere. The origin is still an isolated equilibrium. If we linearize this system around the origin, we would again see a center of the associated linear system. Indeed, for

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} F(x,y) \\ G(x,y) \end{bmatrix}, \quad D\mathbf{f}(\mathbf{0}) = \begin{bmatrix} -3x^2 & -1 \\ 1 & -3y^2 \end{bmatrix} \Bigg|_{\mathbf{x}=\mathbf{0}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

However, the origin in the nonlinear system is, in fact, a *sink*! Now this is difficult to see with the tools that we currently have developed to date, and a quick look using a numerical tool like JOde or the Bluffton University website would reveal something that looks like a sink from far away from the origin, but a center close by the origin. There are a couple of reasons for this:

(1) Numerical models are notorious for behaving poorly in nonstandard situations, as they are approximators and cannot see well subtle behavior, and
(2) The spiraling that is actually in the phase portrait gets very slight as trajectories approach the origin. That is due to the fact that linearly speaking, the phase portrait does look more and more like a center near $\mathbf{0}$.

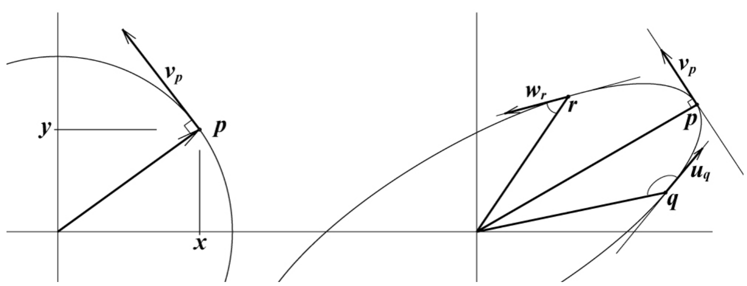
However, this last point is the point. The origin of the nonlinear system is a sink. We just need a proper tool to show this.

Here is some geometry. Given an ODE system

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} F(x,y) \\ G(x,y) \end{bmatrix} = \mathbf{f}(\mathbf{x}),$$

at any point in the phase plane $\mathbf{p} = \begin{bmatrix} x \\ y \end{bmatrix}$, the trajectory through $\mathbf{p}$ will have a tangent vector $\mathbf{v_p} = \begin{bmatrix} F(\mathbf{p}) \\ G(\mathbf{p}) \end{bmatrix}$. The ODE system allows us to calculate with this tangent vector even without solving the system; $\mathbf{v_p}$ is the slope field element at $\mathbf{p}$. Note that here, $\mathbf{p}$ is a point in the phase space. However, $\mathbf{p}$ also represents a vector based at the origin whose head is a $\mathbf{p}$. We can use this to study whether trajectories are getting closer to the origin over time or moving farther away (or neither).

First, suppose that we had a center at the origin, and the trajectories were all true circles centered there. Then the dot product of the vector $\mathbf{p}$ with the slope field element $\mathbf{v_p}$ would be 0 for all choices of $\mathbf{p}$. This is because the tangent vector to the circle $\mathbf{v_p}$ at any point is always perpendicular to the vector from the origin to the circle at that point. That is, $\mathbf{p} \cdot \mathbf{v_p} = \begin{bmatrix} x \\ y \end{bmatrix} \cdot \mathbf{v_p} = 0$. See figure below at left.



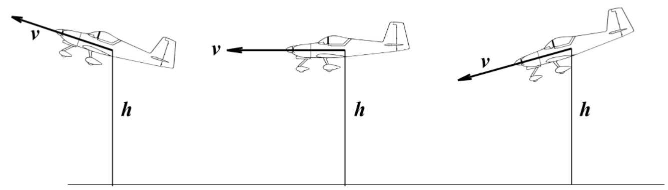
Now consider the case where the trajectory is an ellipse (also a possibility for a center). Then the dot product of a tangent vector to an ellipse and the position vector on the ellipse will sometimes be greater than 0, sometimes equal to 0 (on the axes of the ellipse), and sometimes less than 0. On average, however, it will be 0 (if you are adept at vector calculus, integrate the dot product of the two vectors over the curve of the ellipse and see that it is 0). As in the figure above at right, we see that

$$\mathbf{p} \cdot \mathbf{v_p} = 0, \quad \mathbf{r} \cdot \mathbf{w_r} < 0, \quad \text{and} \quad \mathbf{q} \cdot \mathbf{u_q} > 0.$$

One way to see that is that the slope field vector either has a component in the direction of the position vector *opposite in direction* to the position vector (at $\mathbf{r}$), *in the same direction* as the position vector (at $\mathbf{q}$), or *no component in the direction* of the position vector (at $\mathbf{p}$).

Here is a situation that you may find helpful here (using the figure below): Take an airplane flying over a flat terrain. If the airplane is flying level, maintaining altitude but not climbing, then its trajectory vector $\mathbf{v}$ at an instant is perpendicular to the ground. Constructing a height vector $\mathbf{h}$ from the ground to the plane, the dot product $\mathbf{v} \cdot \mathbf{h} = 0$ since they are perpendicular. This is depicted in the figure below, at center. If the plane is climbing, gaining altitude, moving away from the ground, then $\mathbf{v} \cdot \mathbf{h} > 0$. There is a vertical component of the $\mathbf{v}$ vector, in this case, that is positive with respect to the $\mathbf{h}$ vector. This is shown below at left. If the plane is reducing altitude, getting closer to the ground, then

$\mathbf{v} \cdot \mathbf{h} < 0$. This is because there is a vertical component of $\mathbf{v}$ that is negative with respect to $\mathbf{h}$, and is shown below at right.

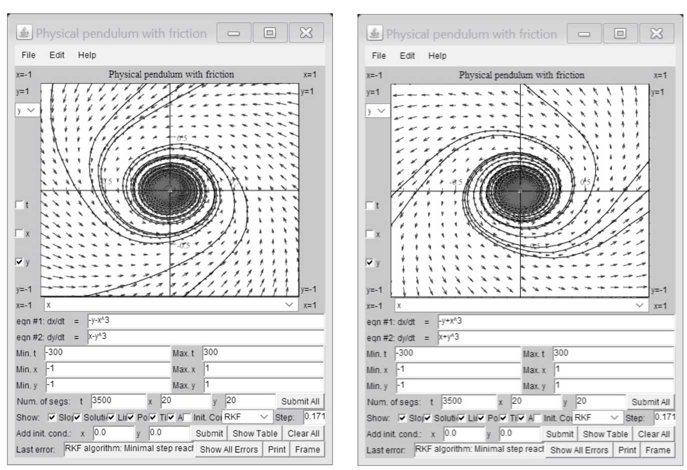


This works in our case also, as we can use the dot product of a trajectory's tangent vector $\mathbf{v_p}$ at a point $\mathbf{p}$ with the vector $\mathbf{p}$ (based at the origin and ending at $\mathbf{p}$) to see if the trajectory is getting closer to the origin (or farther away) as it circles around. If a trajectory is always getting closer to the origin (if the dot product is negative for all positive time), then we know that the trajectory is spiraling inward and its limit, as $t \to \infty$, will be the origin. If this happens for all $\mathbf{p} \neq \mathbf{0}$, then this will be enough to show that the origin is a sink rather than a center.

To this end, take any $\mathbf{p} = \begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \mathbf{0}$. For our system, the slope field elements are $\mathbf{v_p} = \begin{bmatrix} F(\mathbf{p}) \\ G(\mathbf{p}) \end{bmatrix} = \begin{bmatrix} -y - x^3 \\ x - y^3 \end{bmatrix}$. Then

$$\mathbf{p} \cdot \mathbf{v_p} = \begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} -y - x^3 \\ x - y^3 \end{bmatrix} = -xy - x^4 + xy - y^4 = -(x^4 + y^4) < 0.$$

Thus every trajectory is always "reducing altitude" toward $\mathbf{0}$. This is enough to verify that the origin is a sink. See the figure below at left.



And finally, the same analysis can be applied to the almost linear system

$$\begin{aligned} \dot{x} &= -y + x^3 &= F(x, y) \\ \dot{y} &= x + y^3 &= G(x, y) \end{aligned}$$

to show that the origin is a source here, but the linearized system has a center here also. See the above figure at right.