EXAMPLE: PROBLEM 3.6.17 OF THE TEXT

110.302 DIFFERENTIAL EQUATIONS
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Problem. Find the general solution to

$$x^2y'' - 3xy' + 4y = x^2 \ln x, \quad x > 0,$$

given that $y_1(x) = x^2$ and $y_2(x) = x^2 \ln x$ form a fundamental set of solutions to the homogeneous version $x^2y'' - 3xy' + 4y = 0$.

Strategy. We use the method of Variation of Parameters with $Y(x) = u_1(x)x^2 + u_2(x)x^2 \ln x$. To do this, we will need to place the ODE in its standard form to retrieve the non-homogeneous part, $g(t)$. Here, the standard form is $y'' - \frac{3}{x}y' + \frac{4}{x^2}y = \ln x$. So $g(x) = \ln x$.

Solution. With the assumed form for $Y(x)$, we get the system

$$u_1'x^2 + u_2'x^2 \ln x = 0 \quad \text{or} \quad u_1' + u_2' \ln x = 0 \tag{1}$$

$$u_1'2x + u_2'(2x \ln x + x) = \ln x \quad \text{or} \quad 2u_1' + u_2'(1 + 2\ln x) = \frac{\ln x}{x}. \tag{2}$$

Using the simplified system at right, we multiply Equation 1 by -2 and add to Equation 2 and get

$$u_2'\left(-2\ln x + 1 + 2\ln x\right) = \frac{\ln x}{x}, \text{ or } u_2' = \frac{\ln x}{x}, \text{ so } u_2 = \frac{1}{2}(\ln x)^2.$$

Substituting this back into Equation 1, we get

$$u_1' + u_2' \ln x = 0 = u_1' + \left(\frac{\ln x}{x}\right)\ln x, \text{ or } u_1' = -\frac{(\ln x)^2}{x}, \text{ so } u_1 = -\frac{1}{3}(\ln x)^3.$$

Thus our particular solution to the ODE is

$$\begin{aligned} Y(x) &= u_1(x)y_1(x) + u_2(x)y_2(x) \\ &= -\frac{1}{3}(\ln x)^3x^2 + \frac{1}{2}(\ln x)^2x^2 \ln x \\ &= \frac{1}{6}x^2(\ln x)^3. \end{aligned}$$

And our general solution to the ODE is

$$y(x) = c_1x^2 + c_2x^2 \ln x + \frac{1}{6}x^2(\ln x)^3.$$

We can also have done this directly using the integrals constructed in the book. To see this, first we calculate the Wronskian of the two homogeneous solutions:

$$W(y_1, y_2) = W(x^2, x^2 \ln x) = \begin{vmatrix} x^2 & x^2 \ln x \\ 2x & 2x \ln x + x \end{vmatrix} = 2x^3 \ln x + x^3 - 2x^3 \ln x = x^3,$$

which is never zero on the interval $x > 0$.

Then

$$u_1(x) = \int \frac{-y_2 g}{W(y_1, y_2)}\, dx = -\int \frac{(x^2 \ln x)(\ln x)}{x^3}\, dx = -\int \frac{(\ln x)^2}{x}\, dx = -\frac{1}{3}(\ln x)^3,$$

and

$$u_2(x) = \int \frac{y_1 g}{W(y_1, y_2)}\, dx = -\int \frac{x^2 \ln x}{x^3}\, dx = \int \frac{\ln x}{x}\, dx = \frac{1}{2}(\ln x)^2.$$

The rest of the result follows.