Question 1. Solve \( y'' + 9y = 9 \sec^2 3t \) on the interval \( 0 < t < \frac{\pi}{6} \).

**Strategy.** We solve the corresponding homogeneous ODE \( y'' + 9y = 0 \) for the fundamental set of solutions involving \( y_1(t) \) and \( y_2(t) \). Then we use the method of Variation of Parameters with \( Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) \) to find a particular solution to the original, nonhomogeneous ODE.

**Solution.** Here, it is readily apparent that a fundamental set of solutions to \( y'' + 9y = 0 \) is 
\[
 c_1 \cos 3t + c_2 \sin 3t.
\]
Indeed, the characteristic equation of this homogeneous, second-order, linear ODE with constant coefficients is \( r^2 + 9 = 0 \), with solutions \( r = \pm 3i \). With the real part 0 and the imaginary part 3, the result follows.

Now, with this assumed form for \( Y(t) \), we get the system
\[
\begin{align*}
 u_1' \cos 3t + u_2' \sin 3t &= 0 \quad (1) \\
 -3u_1' \sin 3t + 3u_2' \cos 3t &= 9 \sec^2 3t. \quad (2)
\end{align*}
\]
Upon multiplying the first equation by \( 3 \sin 3t \) and the second by \( \cos 3t \) and adding, we obtain the new equation
\[
3u_2' \sin^2 3t + 3u_2' \cos^2 3t = 9 \sec^2 3t \cos 3t = 9 \sec 3t.
\]
Thus \( u_2' = 3 \sec 3t \) so that
\[
 u_2(t) = \ln |\sec 3t + \tan 3t| + C,
\]
although we will ignore the constant.

Substituting this back into Equation 1, we get
\[
 u_1' \cos 3t + u_2' \sin 3t = 0 = u_1' \cos 3t + 3 \sec 3t \sin 3t, \quad \text{or} \quad u_1' = -\frac{3 \sin 3t}{\cos^2 3t} = -3 \tan 3t \sec 3t.
\]
Thus, \( u_1(t) = -\sec 3t \).

Thus our particular solution to the ODE is
\[
 Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) = -\sec 3t \cos 3t + \ln |\sec 3t + \tan 3t| \sin 3t
 = -1 + \ln |\sec 3t + \tan 3t| \sin 3t.
\]
And our general solution to the ODE is
\[
 y(x) = c_1 \cos 3t + c_2 \sin 3t - 1 + \ln |\sec 3t + \tan 3t| \sin 3t.
\]
Now to check whether this particular solution \( Y(t) \) is correct as a solution to the nonhomogeneous ODE:

With \( Y(t) \) as above, we first calculate the two derivatives:

\[
Y(t) = -1 + \ln |\sec 3t + \tan 3t| \sin 3t,
\]
\[
Y'(t) = 3 \sec 3t \sin 3t + 3 \ln |\sec 3t + \tan 3t| \cos 3t
= 3 \tan 3t + 3 \ln |\sec 3t + \tan 3t| \cos 3t,
\]
\[
Y''(t) = 9 \sec^2 3t + 9 \sec 3t \cos 3t - 9 \ln |\sec 3t + \tan 3t| \sin 3t
= 9 \sec^2 3t + 9 - 9 \ln |\sec 3t + \tan 3t| \sin 3t.
\]

Now, we place these derivatives back into the original ODE and see

\[
Y''(t) + 9Y(t) = 9 \sec^2 3t
\]
\[
(9 \sec^2 3t + 9 - 9 \ln |\sec 3t + \tan 3t| \sin 3t) + 9 (-1 + \ln |\sec 3t + \tan 3t| \sin 3t) = 9 \sec^2 3t
\]
\[
9 \sec^2 3t = 9 \sec^2 3t.
\]