PROBLEM SET 7, QUESTION 4: THE RESONANCE PROBLEM

110.302 DIFFERENTIAL EQUATIONS
PROFESSOR RICHARD BROWN

Question 4. A car supported by a MacPherson strut (shock absorber system) travels on a bumpy road at a constant velocity v. The equation modeling the motion of the car is

$$80\ddot{x} + 10000x = 2500\cos\left(\frac{\pi v t}{6}\right). \tag{1}$$

where x represents the vertical position of the cars axle relative to its equilibrium position, and the basic units of measurement are feet and feet per second (this is actually just an example of a forced, un-damped harmonic oscillator, if that is any help). The constant numbers above are related to the characteristics of the car and the strut. Note that the coefficient of time t (inside the cosine) in the forcing term on the right hand side is a frequency, which in this case is directly proportional to the velocity v of the car.

- **(a)** Find the general solution to this nonhomogeneous ODE. Note that your answer will have a term in it which is a function of v.
- **(b)** Determine the value of v for which the solution is undefined (you should present your final answer in miles per hour, as opposed to feet per second).
- **(c)** For a set of initial values $x(0) = \dot{x}(0) = 0$, graph the solutions for a few values of v near your answer in part **(b)** and not so near. Discuss the differences in these graphs and the importance of the special value of v in part **(b)**. (Hint: This special value of v induces what is called resonance in the car).

Strategy. For part **(a)**, the general solution will be a 2-parameter family of solutions, found by solving the homogeneous version, which has constant coefficients, and adding to that a particular solution to the non-homogeneous version. For this particular solution construction, we will employ the Method of Undetermined Coefficients. We will simply use a bit of algebra to solve part **(b)**. We will construct a particular solution to Equation 1 for part **(c)**, but will leave the graphs and discussion to the reader.

Solution. For part **(a)**, the homogeneous version of Equation 1 is

$$80\ddot{x} + 10000x = 0.$$

The characteristic equation for this ODE is then $80r^2 + 10000 = 0$, which is solved by $r = \pm\sqrt{-125} = \pm 5\sqrt{5}\, i$. Being purely imaginary (and hence complex), we can immediately write out the general solution of this homogeneous, linear constant coefficient second order ODE, and get

$$x(t) = c_1 \cos 5\sqrt{5}t + c_2 \sin 5\sqrt{5}t.$$

The general solution to the original nonhomogeneous ODE is then

$$x(t) = c_1 \cos 5\sqrt{5}t + c_2 \sin 5\sqrt{5}t + X(t),$$

where $X(t)$ is any solution to Equation 1. To find a suitable representative for $X(t)$, we can use the Method of Undetermined Coefficients:

Choose $X(t) = A\cos\left(\frac{\pi\nu t}{6}\right)$. We would normally also include another constant involving the sine function, but since there is no first derivative in the ODE, there will not be a sine component (Stop here for a minute to absorb and accept this idea).

Now since $\ddot{X}(t) = -A\left(\frac{\pi\nu}{6}\right)^2 \cos\left(\frac{\pi\nu t}{6}\right)$, we can sub these into the ODE to get

$$\begin{aligned}
80\left(-A\left(\frac{\pi\nu}{6}\right)^2\cos\left(\frac{\pi\nu t}{6}\right)\right) + 10000\left(A\cos\left(\frac{\pi\nu t}{6}\right)\right) &= 2500\cos\left(\frac{\pi\nu t}{6}\right) \\
\left(10000 - 80\left(\frac{\pi\nu}{6}\right)^2\right)A\cos\left(\frac{\pi\nu t}{6}\right) &= 2500\cos\left(\frac{\pi\nu t}{6}\right) \\
A\cos\left(\frac{\pi\nu t}{6}\right) &= \frac{2500}{10000 - 80\left(\frac{\pi\nu}{6}\right)^2}\cos\left(\frac{\pi\nu t}{6}\right) \\
A &= \frac{2500}{10000 - 80\left(\frac{\pi\nu}{6}\right)^2}.
\end{aligned}$$

Hence the general solution to Equation 1, the answer to part **(a)**, is

$$x(t) = c_1 \cos 5\sqrt{5}t + c_2 \sin 5\sqrt{5}t + \frac{2500}{10000 - 80\left(\frac{\pi\nu}{6}\right)^2}\cos\left(\frac{\pi\nu t}{6}\right).$$

You can see how the speed of the car would affect the solution. In fact, there is a special value of ν for which this solution is NOT defined. That is, when $10000 - 80\left(\frac{\pi\nu}{6}\right)^2 = 0$. This is solved by

$$\begin{aligned}
10000 &= 80\left(\frac{\pi\nu}{6}\right)^2 \\
\sqrt{\frac{10000}{80}} &= \frac{\pi\nu}{6} \\
\frac{6\sqrt{125}}{\pi} &= \nu \cong 21.35 \text{ feet per second} \cong 14.55 \text{ miles per hour.}
\end{aligned}$$

This answers part **(b)**.

For part **(c)**, first we find the particular solution corresponding to the initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$. For expediency, let's start with the second one. We calculate:

$$\begin{aligned}
\dot{x}(0) &= \frac{d}{dt}\left[c_1 \cos 5\sqrt{5}t + c_2 \sin 5\sqrt{5}t + \frac{2500}{10000 - 80\left(\frac{\pi\nu}{6}\right)^2}\cos\left(\frac{\pi\nu t}{6}\right)\right]\Bigg|_{t=0} \\
&= \left(-5\sqrt{5}c_1 \sin 5\sqrt{5}t + 5\sqrt{5}c_2 \cos 5\sqrt{5}t - \left(\frac{\pi\nu}{6}\right)\frac{2500}{10000 - 80\left(\frac{\pi\nu}{6}\right)^2}\sin\left(\frac{\pi\nu t}{6}\right)\right)\Bigg|_{t=0} \\
&= 5\sqrt{5}c_2 \cos 5\sqrt{5}(0) = 0
\end{aligned}$$

which implies that $c_2 = 0$. Thus the solution is now

$$x(t) = c_1 \cos 5\sqrt{5}t + \frac{2500}{10000 - 80\left(\frac{\pi\nu}{6}\right)^2} \cos\left(\frac{\pi\nu t}{6}\right).$$

As for the other initial condition, we get

$$\begin{aligned} x(0) &= c_1 \cos 5\sqrt{5}(0) + \frac{2500}{10000 - 80\left(\frac{\pi\nu}{6}\right)^2} \cos\left(\frac{\pi\nu(0)}{6}\right) \\ &= c_1 + \frac{2500}{10000 - 80\left(\frac{\pi\nu}{6}\right)^2} = 0 \\ \implies \quad c_1 &= \frac{-2500}{10000 - 80\left(\frac{\pi\nu}{6}\right)^2} \end{aligned}$$

Thus our particular solution is

$$\begin{aligned} x(t) &= \left(\frac{-2500}{10000 - 80\left(\frac{\pi\nu}{6}\right)^2}\right) \cos 5\sqrt{5}t + \left(\frac{2500}{10000 - 80\left(\frac{\pi\nu}{6}\right)^2}\right) \cos\left(\frac{\pi\nu t}{6}\right) \\ &= \left(\frac{2500}{10000 - 80\left(\frac{\pi\nu}{6}\right)^2}\right) \left(\cos\left(\frac{\pi\nu t}{6}\right) - \cos 5\sqrt{5}t\right). \end{aligned}$$

As for a discussion, note that the term *resonance* refers to a phenomenon in which a vibrating system or external force drives another system to oscillate with greater amplitude at specific frequencies (See the Wikipedia page for "Resonance"). In our case, the *frequency* of our sinusoidal forcing term depends on the car's velocity ν. But the *solution* to the mathematical model for this set of initial conditions depends on ν also in the amplitude of the function. Recall that the solution is modeling the vertical position of the car's axle relative to it's rest position. So, on this bumpy road, as we slowly increase the car's speed toward this special value of ν, what happens to the car?