THEORY: SOLUTION DOMAINS FOR DIFFERENTIAL EQUATIONS

110.302 DIFFERENTIAL EQUATIONS
PROFESSOR RICHARD BROWN

Here is an example for Page 37 of the 10th edition of the text by Boyce and DiPrima:

Example 1. Solve the Initial Value Problem (IVP) $ty' + 2y^2 = 4t^2$ for the initial conditions

$$\textbf{(a)}\quad y(-1) = 1, \quad \textbf{(b)}\; y(-1) = 2, \quad \textbf{(c)}\; y(0) = 0, \quad \textbf{(d)}\; y(0) = 1.$$

Strategy. This ODE is linear, so we use the integrating factor to solve and present the general solution. Then we find the particular solution for each and check the domain of the resulting function.

Solution. First, we seek to find the general solution to the ODE. Here the ODE in standard form is

$$y' + \frac{2}{t}y = 4t,$$

so that in the standard form $y' + p(t)y = q(t)$, we have the coefficient of y to be $p(t) = \frac{2}{t}$. Then the integrating factor is

$$e^{\int p(t)\,dt} = e^{\int \frac{2}{t}\,dt} = e^{2\ln|t|} = t^2.$$

Multiply this integrating factor by the standard form of the ODE and we get

$$\begin{aligned} t^2\left[y' + \frac{2}{t}y \right.&= \left. 4t\right] \\ t^2y' + 2ty &= 4t^3 \\ \frac{d}{dt}\left[t^2y\right] &= 4t^3. \end{aligned}$$

Now we integrate both sides of this last equation with respect to t and get

$$\begin{aligned} \int \left(\frac{d}{dt}\left[t^2y\right]\right)\,dt &= \int 4t^3\,dt \\ t^2y &= t^4 + C \\ y(t) &= t^2 + \frac{C}{t^2}. \end{aligned}$$

This last equation is the general solution to the ODE.

Now for the initial data:

- For **(a)** $y(-1) = 1$, we have

  $$y(-1) = 1 = (-1)^2 + \frac{C}{(-1)^2},$$

  so that $C = 0$, and the particular solution to this IVP is $y(t) = t^2$, whose domain is all real numbers.

- For **(b)** $y(-1) = 2$, we have
$$y(-1) = 2 = (-1)^2 + \frac{C}{(-1)^2},$$
so that $C = 1$, and the particular solution to this IVP is $y(t) = t^2 + \frac{1}{t}$, whose domain is all $t \in (-\infty, 0)$. (Why this piece?)
- For **(c)** $y(0) = 0$, we cannot plug in $t = 0$ into the general solution. However, the point $(t, y) = (0, 0)$ in the ty-plane is on the curve $y(t) = t^2$, so the particular solution to this IVP is again $y(t) = t^2$, and the domain is all real numbers.
- And lastly, for **(d)** $y(0) = 1$, the point $(t, y) = (0, 1)$ is not on any integral curve. The IVP
$$ty' + 2y = 4t^2, \quad y(0) = 1$$
has no solution.

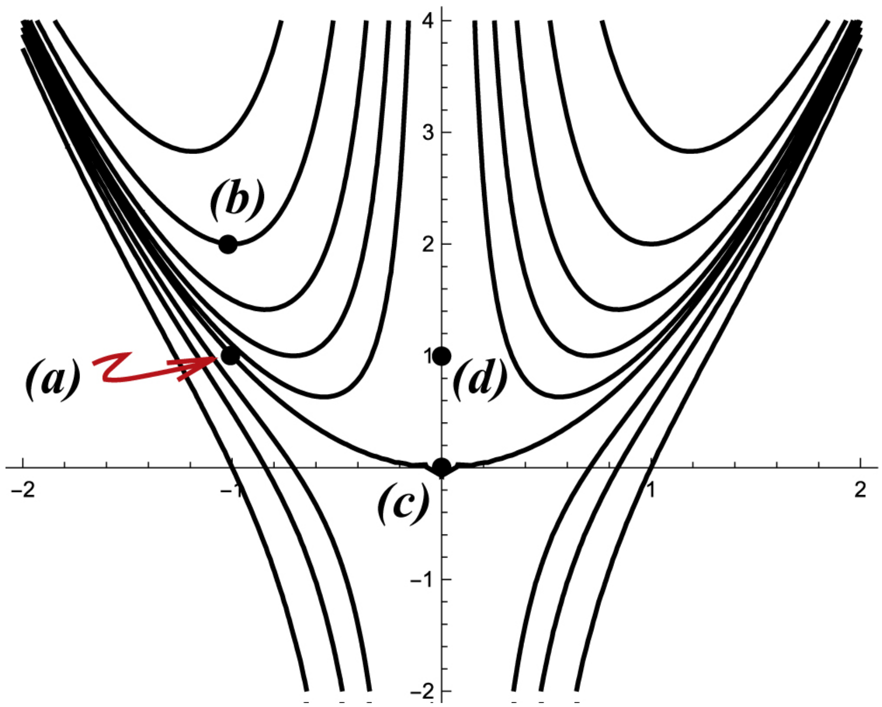

FIGURE 1. Some integral curve of the general solution, along with the initial data.

So what is happening here? The domains for each of these solutions are simply the continuous pieces of the function $y(t) = t^2 + \frac{C}{t^2}$, for different values of C. For almost every value of C, the vertical axis is a vertical asymptote, and outside of the point $(t, y) = (0, 0)$, no points on the vertical axis can serve as initial data in an IVP.