EXAMPLES OF INCORRECT GUESSES IN USING UNDETERMINED COEFFICIENTS FOR LINEAR NONHOMOGENEOUS ODES

110.302 DIFFERENTIAL EQUATIONS
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Given a second order, linear, nonhomogeneous differential equation

$$y'' + p(t)y' + q(t)y = g(t),$$

where $p(t)$, $q(t)$, and $g(t)$ are all continuous on some open t-interval I, we know the following:

- By existence and uniqueness criteria for linear ODEs, solutions will be uniquely defined at all points in I and for all $y \in \mathbb{R}$, and the domain for all solutions inside I is all of I;
- The general solution to this nonhomogeneous ODE can be found by finding a fundamental set of solutions to the homogeneous version of this ODE, $y''+p(t)y'+q(t)y = 0$, and adding to it any particular solution to the nonhomogeneous ODE. Indeed, if $y_1(t)$ and $y_2(t)$ are solutions to the homogeneous version of this ODE, and they are independent (their Wronskian is non-zero), and if $Y(t)$ is any particular solution to the nonhomogeneous version of the ODE, then the general solution to the nonhomogeneous version of the ODE is
$$y(t) = c_1y_1(t) + c_2y_2(t) + Y(t), \quad \text{for} \quad c_c, c_2 \in \mathbb{R}.$$
- Under the two conditions that (1) the homogeneous version of the ODE has constant coefficients, and (2) $g(t)$ is a sum of products of polynomials, sines and cosines, and exponentials, one can construct a guess at a particular solution to the nonhomogeneous ODE via the method of Undetermined Coefficients. This guess will be a function that is known up to some coefficient constants, and formed according to a chart in the text. Once formed, this guess can then be plugged back into the nonhomogeneous ODE and one can then solve for the unknown coefficients. Thus a particular solution can be constructed.

The tricky part of this endeavor involves the correct guess. I mentioned in Lecture 16 two examples where the guess is a bit more involved than a simple identification of function types. Herein, I will expand on that.

Example 1. Solve the ODE $y'' - 4y' + 4y = 12e^{2t}$.

Strategy. Solve the homogeneous ODE $y'' - 4y' + 4y = 0$, and then use the method of Undetermined Coefficients to construct a nonhomogeneous solution to the original ODE. Combine these two to write a general solution to the nonhomogeneous ODE.

Solution. First, the homogeneous ODE $y'' - 4y' + 4y = 0$ has characteristic equation $r^2 - 4r + 4 = 0 = (r-2)^2$ with the single solution $r = 2$. Hence we can immediately write down a fundamental set of solutions to this as $c_1e^{2t} + c_2te^{2t}$.

To construct a solution to the nonhomogeneous ODE, we note that the homogeneous version of the ODE has constant coefficients, and that $g(t) = 12e^{2t}$ is an exponential function. Thus we can use the method of Undetermined Coefficients to construct this nonhomogeneous solution. According to the chart in the text, a proper guess would be

$$Y(t) = At^2e^{2t}, \quad \text{for} \quad A \in \mathbb{R}.$$

Note that we cannot use as a guess either $Y(t) = Ae^{2t}$ or $Y(t) = Ate^{2t}$ since both of these are homogeneous solutions for any choice of A.

To see these latter two guesses will not work, try them:

(1) **Attempt, as a guess, $Y(t) = Ae^{2t}$.**
Then $Y'(t) = 2Ae^{2t}$ and $Y''(t) = 4Ae^{2t}$. Plugging this into the ODE yields

$$\begin{aligned} Y''(t) - 4Y'(t) + 4Y(t) &= 12e^{2t} \\ 4Ae^{2t} - 4\left(2Ae^{2t}\right) + 4Ae^{2t} &= 12e^{2t} \\ 4Ae^{2t} - 8Ae^{2t} + 4Ae^{2t} &= 12e^{2t} \\ 0 &= 12e^{2t}, \end{aligned}$$

which is an *inconsistent equation*, which means that it has no solutions, or does not make sense, or cannot be satisfied. Hence the guess was not of the proper form to work.

(2) **Attempt, as a guess, $Y(t) = Ate^{2t}$.**
Then $Y'(t) = Ae^{2t} + 2Ate^{2t}$ and $Y''(t) = 2Ae^{2t} + 2Ae^{2t} + 4Ate^{2t}$. Plugging this into the ODE yields

$$\begin{aligned} Y''(t) - 4Y'(t) + 4Y(t) &= 12e^{2t} \\ \left(4Ae^{2t} + 4Ate^{2t}\right) - 4\left(Ae^{2t} + 2Ate^{2t}\right) + 4Ate^{2t} &= 12e^{2t} \\ 4Ae^{2t} + 4Ate^{2t} - 4Ae^{2t} - 8Ate^{2t} + 4Ate^{2t} &= 12e^{2t} \\ 0 &= 12e^{2t}, \end{aligned}$$

which is, again, an inconsistent equation. Hence the guess was, yet again, not of the proper form to work.

So we try what should be a good guess: $Y(t) = At^2e^{2t}$.

Then $Y'(t) = 2Ate^{2t} + 2At^2e^{2t}$ and $Y''(t) = 2Ae^{2t} + 4Ate^{2t} + 4Ate^{2t} + 4At^2e^{2t}$. Plugging this into the ODE yields

$$\begin{aligned} Y''(t) - 4Y'(t) + 4Y(t) &= 12e^{2t} \\ \left(2Ae^{2t} + 8Ate^{2t} + 4At^2e^{2t}\right) - 4\left(2Ate^{2t} + 2At^2e^{2t}\right) + 4At^2e^{2t} &= 12e^{2t} \\ 2Ae^{2t} + 8Ate^{2t} + 4At^2e^{2t} - 8Ate^{2t} - 8At^2e^{2t} + 4At^2e^{2t} &= 12e^{2t} \\ 2Ae^{2t} &= 12e^{2t} \\ 2A &= 12, \quad \text{or} \quad A = 6. \end{aligned}$$

This equation is consistent and renders the unique solution $A = 6$. Thus $Y(t) = 6t^2e^{2t}$ is a nonhomogeneous solution to the original nonhomogeneous ODE, and we can form the general solution to the ODE:

$$y(t) = c_1e^{2t} + c_2te^{2t} + 6t^2e^{2t}.$$

Example 2. Solve the ODE $y'' - 4y' + 4y = 3t^3e^{2t}$.

Strategy. Solve the homogeneous ODE $y'' - 4y' + 4y = 0$, and then use the method of Undetermined Coefficients to construct a nonhomogeneous solution to the original ODE. Combine these two to write a general solution to the nonhomogeneous ODE.

Solution. As in the last example, a fundamental set of solutions to the homogeneous ODE $y'' - 4y' + 4y = 0$ is $c_1e^{2t} + c_2te^{2t}$. Also, $g(t) = 2t^3e^{2t}$ is a product of a polynomial and an exponential, so we can use the method of Undetermined Coefficients here to construct our guess.

According to the structure of what we consider a valid guiess at a solution, a valid guess should be:

$$Y(t) = t^2\left(At^3 + Bt^2 + Ct + D\right)e^{2t}.$$

Some reasons behind this guess are the following:

- When a polynomial is present in $g(t)$, then the guess must include all monomials of degree less that the degree of the polynomial, where each of those coefficients is unknown.
- If any part of the guess includes summands that are homogeneous solutions, then one must account for that by the extra term t^s, where s is the smallest positive integer that renders all summands of a solution independent of the homogeneous solutions. Here, since both e^{2t} and te^{2t} are homogeneous solutions, we need to tack on the factor t^2 to our guess.

As before, let's look at some incorrect guesses and see that they fail and why:

(1) **Attempt, as a guess, $Y(t) = At^3e^{2t}$.**
Note that this guess may seem reasonable, since t^3e^{2t} is independent of both the homogeneous solutions e^{2t} and te^{2t} (check the Wronskians). So we give it a shot! Here

$$\begin{aligned} Y'(t) &= 3At^2e^{2t} + 2At^3e^{2t}, \quad \text{and} \\ Y''(t) &= 6Ate^{2t} + 6At^2e^{2t} + 6At^2e^{2t} + 4At^3e^{2t} \\ &= 6Ate^{2t} + 12At^2e^{2t} + 4At^3e^{2t}. \end{aligned}$$

Plugging this into the nonhomogeneous ODE yields

$$\begin{aligned} Y''(t) - 4Y'(t) + 4Y(t) &= 3t^3e^{2t} \\ \left(6Ate^{2t} + 12At^2e^{2t} + 4At^3e^{2t}\right) - 4\left(3At^2e^{2t} + 2At^3e^{2t}\right) + 4At^3e^{2t} &= 3t^3e^{2t} \\ 6Ate^{2t} + 12At^2e^{2t} + 4At^3e^{2t} - 12At^2e^{2t} - 8At^3e^{2t} + 4At^3e^{2t} &= 3t^3e^{2t} \\ 6Ate^{2t} &= 3t^3e^{2t}. \end{aligned}$$

But $A \in \mathbb{R}$ is a constant, and the simplified version of this last equation $2A = t^2$ must hold true for all $t \in \mathbb{R}$ in this case, since solutions exist and are uniquely defined on all $\mathbb{R}$ by the existence and uniqueness criteria. As A here is a function of t, the method fails with this guess. What is wrong here is that we did not account for the lower degree terms in our guess polynomial. So let's try a new guess:

(2) **Attempt, as a guess,** $Y(t) = \left(At^3 + Bt^2 + Ct + D\right)e^{2t}$**.**
Again, this seems reasonable since we will now account for the lower degree terms in the derivatives. We have

$$\begin{aligned} Y'(t) &= \left(3At^2 + 2Bt + C\right)e^{2t} + 2\left(At^3 + Bt^2 + Ct + D\right)e^{2t}, \quad \text{and} \\ Y''(t) &= (6At + 2B)\,e^{2t} + 2\left(3At^2 + 2Bt + C\right)e^{2t} + 2\left(3At^2 + 2Bt + C\right)e^{2t} \\ &\quad + 4\left(At^3 + Bt^2 + Ct + D\right)e^{2t} \\ &= (6At + 2B)\,e^{2t} + 4\left(3At^2 + 2Bt + C\right)e^{2t} + 4\left(At^3 + Bt^2 + Ct + D\right)e^{2t}. \end{aligned}$$

Placed back into the ODE, we get

$$\begin{aligned} Y''(t) - 4Y'(t) + 4Y(t) &= 3t^3e^{2t} \\ \Big((6At + 2B)\,e^{2t} + 4\left(3At^2 + 2Bt + C\right)e^{2t} + 4\left(At^3 + Bt^2 + Ct + D\right)e^{2t} \Big) & \\ -4\Big(\left(3At^2 + 2Bt + C\right)e^{2t} + 2\left(At^3 + Bt^2 + Ct + D\right)e^{2t} \Big) & \\ +4\left(At^3 + Bt^2 + Ct + D\right)e^{2t} &= 3t^3e^{2t}. \end{aligned}$$

This renders four separate equations in the coefficients, one for each of the terms t^se^{2t} for $s = 0, 1, 2, 3$.

e^{2t}**-eqn:** $2B + 4C + 4D - 4C - 8D + 4D = 0$. Hence $B = 0$.
te^{2t}**-eqn:** $6A + 8B + 4C - 8B - 8C + 4C = 0$. Hence $A = 0$.
t^2e^{2t}**-eqn:** $12A + 4B - 12A - 8B + 4B = 0$. Hence this equation is useless to us.
t^3e^{2t}**-eqn:** $4A - 8A + 4A = 3$. And we have an inconsistent equation $0 = 3$.

At this point, we are done since there are no real constants C and D where a guess like this $(Ct + D)e^{2t}$, together with its derivatives, can ever equal $3t^3e^{2t}$ for all $t \in \mathbb{R}$. Hence we cannot use this guess to produce a particular nonhomogeneous solution.

So we could also try something like $Y(t) = t\left(At^3 + Bt^2 + Ct + D\right)e^{2t}$ also, but we will have the same problem (try it). That is, at least one of the terms in the guess will correspond to a homogeneous solution (Namely Dte^{2t}). To avoid this, we construct our "valid" guess in a fashion that no terms correspond to homogeneous solutions:

Attempt, as a guess, $Y(t) = t^2\left(At^3 + Bt^2 + Ct + D\right)e^{2t} = \left(At^5 + Bt^4 + Ct^3 + Dt^2\right)e^{2t}$**.**
Again, this seems reasonable since at least now all terms in the guess are independent of the

two homogeneous solutions. We have

$$\begin{aligned}
Y'(t) &= \left(5At^4 + 4Bt^3 + 3Ct^2 + 2Dt\right)e^{2t} + 2\left(At^5 + Bt^4 + Ct^3 + Dt^2\right)e^{2t}, \quad \text{and} \\
Y''(t) &= \left(20At^3 + 12Bt^2 + 6Ct + 2D\right)e^{2t} + 2\left(5At^4 + 4Bt^3 + 3Ct^2 + 2Dt\right)e^{2t} \\
&\quad + 2\left(5At^4 + 4Bt^3 + 3Ct^2 + 2Dt\right)e^{2t} + 4\left(At^5 + Bt^4 + Ct^3 + Dt^2\right)e^{2t} \\
&= \left(20At^3 + 12Bt^2 + 6Ct + 2D\right)e^{2t} + 4\left(5At^4 + 4Bt^3 + 3Ct^2 + 2Dt\right)e^{2t} \\
&\quad + 4\left(At^5 + Bt^4 + Ct^3 + Dt^2\right)e^{2t}.
\end{aligned}$$

Placed back into the ODE, we get

$$\begin{aligned}
&Y''(t) - 4Y'(t) + 4Y(t) = 3t^3e^{2t} \\
&\Big(\left(20At^3 + 12Bt^2 + 6Ct + 2D\right)e^{2t} + 4\left(5At^4 + 4Bt^3 + 3Ct^2 + 2Dt\right)e^{2t} \\
&\qquad + 4\left(At^5 + Bt^4 + Ct^3 + Dt^2\right)e^{2t}\Big) \\
&-4\Big(\left(5At^4 + 4Bt^3 + 3Ct^2 + 2Dt\right)e^{2t} + 2\left(At^5 + Bt^4 + Ct^3 + Dt^2\right)e^{2t}\Big) \\
&\qquad +4\left(At^5 + Bt^4 + Ct^3 + Dt^2\right)e^{2t} = 3t^3e^{2t}.
\end{aligned}$$

Now the separate equations for each of the terms in the coefficients become:

e^{2t}-eqn: $2D = 0$.
te^{2t}-eqn: $6C + 8D - 8D = 0$. Hence also $C = 0$.
t^2e^{2t}-eqn: $12B + 12C + 4D - 12C - 8D + 4D = 0$, rendering $B = 0$.
t^3e^{2t}-eqn: $20A - 16B + 4C - 16B - 8C + 4C = 3$. And we now have a non-zero coefficient, $A = \frac{3}{20}$.
t^4e^{2t}-eqn: $20A + 4B - 20A - 8B + 4B = 0$, rendering this equation useless.
t^5e^{2t}-eqn: $4A - 8A + 4A = 0$, again rendering a useless equation.

Hence this guess does render a particular solution, namely

$$Y(t) = t^2\left(\frac{3}{20}t^3 + 0t^2 + 0t + 0\right)e^{2t} = \frac{3}{20}t^5e^{2t}.$$

And finally, we can now construct our general solution to the ODE $y'' - 4y' + 4y = 3t^3e^{2t}$, and it is

$$y(t) = c_1e^{2t} + c_2te^{2t} + \frac{3}{20}t^5e^{2t}.$$