EXAMPLE: THE WRONSKIAN DETERMINANT OF A SECOND-ORDER, LINEAR HOMOGENEOUS DIFFERENTIAL EQUATION

110.302 DIFFERENTIAL EQUATIONS
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Problem. Solve the ODE $2y'' + 8y' - 10y = 0$.

Strategy. Solving this ODE means finding a fundamental set of solutions so that ALL solutions are given by the general form. Noting that this ODE has constant coefficients, we solve this problem by finding two exponential solutions using the characteristic equation, and then show that all solutions are a linear combination of these.

Solution. This ODE is linear with constant coefficients. Written in the form $y'' + p(t)y' + q(t)y = 0$, namely

$$y'' + 4y' - 5y = 0,$$

then with $p(t) = 4$ and $q(t) = -5$ both continuous for all t, we know by existence and uniqueness that solutions will exist and be unique everywhere (on $I = (-\infty, \infty)$). We can construct the characteristic equation directly (by assuming a solution is exponential $y = e^{rt}$, and then solving for the values of r, if they exist), and write

$$r^2 + 4r - 5 = 0 = (r+5)(r-1).$$

Thus, we know that $y_1 = e^t$, and $y_2 = e^{-5t}$ are two solutions (try them out).

So we also know then that (since the ODE is linear) any linear combination of these two solutions is also a solution, so that $y(t) = c_1 y_1(t) + c_2 y_2(t)$ is also a solution for any choice of c_1, $c_2 \in \mathbb{R}$. And since

$$\begin{aligned} W(y_1, y_2)(t) = W(e^t, e^{-5t})(t) &= \begin{vmatrix} e^t & e^{-5t} \\ e^t & -5e^{-5t} \end{vmatrix} \\ &= -5e^t e^{-5t} - e^t e^{-5t} = -6e^{-4t} \neq 0 \end{aligned}$$

for any $t \in \mathbb{R}$, we know by theorems in class that ALL solutions can be written this way and the fundamental set of solutions to the ODE is given by

$$\boxed{y(t) = c_1 e^t + c_2 e^{-5t}}.$$

Let's say, for the sake of argument, that you found two solutions $y_1(t) = e^{-5t}$ and $y_2(t) = 8e^{-5t}$. You can substitute both of these into the ODE and true enough, both are solutions. Hence ANY linear combination will also be a solutions. Hence one can write, as before, $y(t) = c_1 y_1(t) + c_2 y_2(t) = c_1 e^{-5t} + 8c_2 e^{-5t}$. Yeah, I know it does not look right, but let's go with it. Does this constitute a fundamental set of solutions? Can ALL solutions be written this way? The answer is no, and we can show this in two ways: (1) Check the Wronskian determinant:

$$W(y_1, y_2)(t) = W(e^{-5t}, 8e^{-5t})(t) = \begin{vmatrix} e^{-5t} & 8e^{-5t} \\ -5e^{-5t} & -40e^{-5t} \end{vmatrix}$$
$$= -40e^{-5t}e^{-5t} + 40e^{-5t}e^{-5t} = 0e^{-10t} = 0$$

everywhere. The Wronskian determinant indicates that these two solutions are NOT sufficiently different, and DO NOT make a fundamental set of solutions.

The second way is, (2) notice that with our new choice of two solutions, we get

$$y(t) = c_1y_1(t) + c_2y_2(t) = c_1e^{-5t} + 8c_2e^{-5t} = (c_1 + 8c_2)e^{-5t} = Ke^{-5t}.$$

Really, there is only one exponential in this expression. Knowing that the function $y(t) = e^t$ also solves the ODE, if we tried to use $y(t) = c_1e^{-5t} + 8c_2e^{-5t} = Ke^{-5t}$ as our fundemental set of solutions, then there would have to be a value for K so that

$$y(t) = e^t = Ke^{-5t}.$$

Can there be such a value for K? There cannot. Try to show there is a value for K that will work for all $t \in \mathbb{R}$.

We can go one step further:

Problem. Solve the IVP $2y'' + 8y' - 10y = 0$, $y(0) = 2$, and $y'(0) = 3$.

Strategy. We use all of the above analysis to answer this question, using both the correct way and the latter incorrect stuff.

Solution. The fundamental set of solutions to this ODE was given above as $y(t) = c_1e^t + c_2e^{-5t}$. Using the initial data, we get

$$y(0) = c_1e^0 + c_2e^{-5(0)} = c_1 + c_2 = 2$$
$$y'(0) = c_1e^0 - 5c_2e^{-5(0)} = c_1 - 5c_2 = 3.$$

Solving for c_1 and c_2, we get $c_1 = \frac{13}{6}$ and $c_2 = -\frac{1}{6}$, and our solution to the IVP is

$$\boxed{y(t) = \frac{13}{6}e^t - \frac{1}{6}e^{-5t}}.$$

You should check that this indeed solves the IVP.

Let's try to use the set of solutions $y(t) = c_1y_1(t) + c_2y_2(t) = c_1e^{-5t} + 8c_2e^{-5t}$ as our fundamental set of solutions. Then we will get

$$y(0) = c_1e^{-5(0)} + 8c_2e^{-5(0)} = c_1 + 8c_2 = 2$$
$$y'(0) = -5c_1e^{-5(0)} - 40c_2e^{-5(0)} = -5c_1 - 40c_2 = 3.$$

Multiply the first equation by 5 and then add the two equations to get $0 = 13$. Thus there are no solutions to this system of two linear equations in two unknowns. Thus THERE IS NO set of values for c_1 and c_2, so that $y(t) = c_1e^{-5t} + 8c_2e^{-5t}$ solves the IVP. Since we definitely know that this ODE does have solutions everywhere, the ONLY thing that must be wrong is our assumption that we have a fundamental set of solutions.

Basically, the "other" function e^t must be able to play a role in constructing solutions. Does this make sense?