EXAMPLE: THE WRONSKIAN DETERMINANT OF A SECOND-ORDER, LINEAR HOMOGENEOUS DIFFERENTIAL EQUATION

110.302 DIFFERENTIAL EQUATIONS
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Given a second order, linear, homogeneous differential equation

$$y'' + p(t)y' + q(t)y = 0,$$

where both $p(t)$ and $q(t)$ are continuous on some open t-interval I, and two solutions $y_1(t)$ and $y_2(t)$, one can form a fundamental set of solutions as the linear combination of these two

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

ONLY under the condition that the Wronskian determinant

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} \neq 0$$

for all $t \in I$. This condition implies that the two differentiable functions $y_1(t)$ and $y_2(t)$ are independent; that there does not exist two constants $k_1, k_2 \in \mathbb{R}$, not both equal to 0, where $k_1 y_1(t) + k_2 y_2(t) = 0$ on I. Then we know that ALL solutions to the ODE will be a linear combination of $y_1(t)$ and $y_2(t)$.

We can actually take this further. It turns out that in this situation, and on the interval I, either of two things can happen:

- The Wronskian is always 0 on I (we say $W(y_1, y_1)$ is identically 0 on I, or $W(y_1, y_2) \equiv 0$ on I), or
- the Wronskian is NEVER 0 on I.

Why? Because of the following, established by Niels Henrik Abel in the early 1800's:

Theorem (Abel's Theorem)**.** *If $y_1(t)$ and $y_2(t)$ are two solutions to the ODE $y'' + p(t)y' + q(t)y = 0$, where $p(t)$ and $q(t)$ are continuous on some open t-interval I, then*

$$W(y_1, y_2)(t) = Ce^{-\int p(t)\, dt}$$

where C depends on the choice of y_1 and y_2, but not on t.

First, some notes:

- If $y_1(t)$ and $y_2(t)$ are linearly dependent, then $C = 0$.
- If $y_1(t)$ and $y_2(t)$ are linearly independent, then $W(y_1, y_2) \neq 0$ on ALL of I.
- Linear independence and non-zero Wronskian are the same thing for solutions to these ODEs.

Proof. Since y_1 and y_1 both solve the ODE, we have

$$\begin{aligned} y_1'' + p(t)y_1' + q(t)y_1 &= 0 \\ y_2'' + p(t)y_2' + q(t)y_2 &= 0. \end{aligned}$$

Here, multiply the first equation by $-y_2$ and the second by y_1 and then add them together (this will eliminate the coefficient $q(t)$)

$$\underbrace{(y_1y_2'' - y_2y_1'')}_{W'(y_1,y_2)} + p(t)\underbrace{(y_1y_2' - y_2y_1')}_{W(y_1,y_2)} = 0,$$

Which leads to a first order differential equation whose variable IS the Wronskian determinant itself (really, the Wronskian is a function of t)

$$W' + p(t)W = 0.$$

This first order ODE is both linear and separable, and by separation of variables, we get

$$\frac{W'}{W} = -p(t) \quad \Longrightarrow \quad \ln|W| = -\int p(t)\,dt + K \quad \Longrightarrow \quad Ce^{-\int p(t)\,dt}.$$

□

Hence by the notes above just before the proof, either $C = 0$, and the Wronskian is always 0, and the two solutions are linearly dependent, or $C \neq 0$, and the Wronskian is NEVER 0, and the two solutions are linearly independent.