Lecture 1:

Start with \( y = f(x) \), some unknown functional relation between 2 variables, where
- \( x \) - independent variable
- \( y \) - dependent variable

Such an equation (or sets of them are) called a mathematical model when the variables represent measurable quantities in some application (usually set up to study some unknown entity \( y \) based on its relationship to something controllable \( x \)).

If \( y = f(x) \) is known, then we can simply study its properties (using calculus).

Often, though, we do not know \( y = f(x) \), but we do have information about some properties, like derivatives, for example.
examples

I  \( \frac{dy}{dx} = ky \), \( k \in \mathbb{R} \).

II  \( F = ma \) (Newton's 2nd Law of Motion).

III  \( f'(x) = x - e^{2x} \) (restatement of:

Find \( \int (k - e^{2x}) \, dx \).

or \( y' = g(x) \), \( y(x) \) is unknown soln: \( y(x) = \int g(x) \, dx \).

IV  \( \frac{d^2 \theta}{dt^2} + \frac{g}{L} \sin \theta = 0 \)

The Pendulum.

The above are mathematical models where the actual function is only known implicitly....

Def. An ordinary differential equation (ODE) is an equation involving an unknown function between 2 variables and some of its derivatives.

Note: "Ordinary" means that the unknown function is a function of one independent variable.
ex. The Heat Equation (in 3-space)

\[ \frac{dx}{dt} = \alpha \left( \frac{\partial^2 x}{\partial y^2} + \frac{\partial^2 x}{\partial z^2} \right) \]

is a partial differential equation since \( x \) is a function of more than one independent variable.

Def. The order of an ODE is the same as the order of the highest derivative that appears in the equation.

Ex. (i) and (ii) are first order ODEs.
   (iii) and (iv) are 2nd order ODEs (do you see this?)

Def. The general form of an n-th order ODE is

\[ F(x, y, y', \ldots, y^{(n)}) = 0 \quad (\ast) \]

where \( x \) is independent (time?)

- \( y \)-dependent (unknown function \( y = f(x) \))

- \( y^{(i)} \) is the \( i \)-th derivative of \( y = f(x) \)

- \( F \) is some expression in \( x, y, y', \ldots, y^{(n)} \)
Note: Sometimes, we can solve for the highest derivative:

\[ y^{(n)} = \lambda(x, y, y', \ldots, y^{(n-1)}) \]

But not always:

ex. \[ y^{(1)} + \sin y^{(2)} = y^{(3)} \] cannot be written

like \( (x, y) \). But for \( (x) \), \( F = y^{(1)} + \sin y - y^{(3)} \).

**Def.** A function \( F(x_1, \ldots, x_n, y_1, \ldots, y_m) \) is linear in the variables \( y_1, \ldots, y_m \) if

\[ F(x_1, \ldots, x_n, y_1, \ldots, y_m) = \lambda_0(x) + \sum_{i=1}^{m} \lambda_i(x) y_i \]

where the \( \lambda_i(x) \), \( i = 0, \ldots, m \) are arbitrary.

Notes:

1. For an ODE to be linear, it must be linear in \( y, y', \ldots, y^{(n)} \), and can be written as

\[ \lambda_n(x) y^{(n)} + \ldots + \lambda_1(x) y' + \lambda_0(x) y = g(x) \]
Examples

A. \((\sin x) y' + (\ln x) y = e^x\) is a linear \(1^{st}\) order ODE.

B. \(y'' + xy' + \sin y = 0\) is non-linear.

C. \(y'' y + y' = 0\) is not linear.

Suppose \(y' = f(t, y)\) is a \(1^{st}\) order linear ODE (written like \((xfx)\)).

Then there exist functions \(p(t), q(t)\) so that
\(f(t, y) = p(t)y + q(t)\), and the ODE can be written as
\[ y' = p(t)y + q(t) \]

This form will be very important in understanding how to study this type of ODE.

Ex. Given \((\sin x) y' + (\ln x) y = \tan x e^x\), identify \(p(t)\).

Solution: Divide by \(\sin x\) to set,
\[ y' + \frac{(\ln x)}{\sin x} y = \frac{\tan x e^x}{\sin x}. \quad p(t) = \frac{\ln x}{\sin x}. \]
A solution to (10) on \( I = (a,b) \subset \mathbb{R} \)

is any function \( y = f(x) \) that satisfies

the equation \((10)\).\)

\[ \frac{dp}{dt} = \frac{p}{2} - 450 \quad \text{is solved by} \quad p(t) = 900 + Ce^{\frac{t}{2}} \]

\( \forall C \in \mathbb{R} \).\)

- How do we know? Try it.
- How did we find it? Keep listening.
- What do we make of the parameter \( C \)?

Sometimes a solution is only known implicitly:

Show \( x^2 + y^2 - 5 = 0 \) solves \( \frac{dy}{dx} = \frac{-x}{y} \).

Here only locally can we solve for \( y(x) \).

- Solve \( x''(t) + \frac{k}{m} x(t) = 0 \).
- Solve \( y'(x) = x - e^{xy} \).