110.302 Lecture 2:

Compare the following:

ⓐ $\frac{dy}{dx} = x - e^{x/2}$ ⓑ $\frac{dp}{dt} = \frac{p}{2} - 450$

① Both are First order ODES.

② Both are linear (can you see this?)

③ ⓐ is of the form $y' = f(x)$ ← independent variable on RHS

and is simply a calculus problem

(think: Find $\int f(x)\,dx$)

Here the RHS is only a fnc of the independent variable.

- Integrate both sides with respect to x to get

$$y(x) = \frac{x^2}{2} - 2e^{x/2} + C$$

- This formula is called a general solution to ⓐ as it is an expression that specifies all possible solutions.

Integrate the equation. What does that mean?

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- A particular solution to ⓐ involves choosing a value for the constant c.

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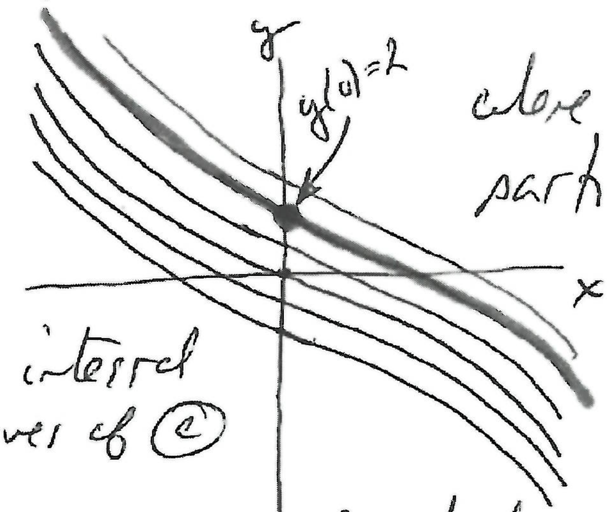
- Graphs of solutions are called integral curves (why?)



above each is a particular solution

Some integral curves of ⓐ

Not in text.

- We also call the general solution to ⓐ a 1-parameter family of solutions.

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With the addition of a single pt in the xy-plane (ex. $y(0)=2$), called an initial value, we can "choose" a particular solution from the family.

With this point, there is now only 1 solution to the problem: (the particular sol.)

An ODE typically has lots of solutions. An IVP only one. We solve the IVP by integration, which creates the unknown constant c.

$$y(x) = \frac{x^2}{2} - 2e^{x/2} + C$$

$$y(0) = \frac{0^2}{2} - 2e^{0/2} + C = 2$$

$$= 0 - 2 + C = 2 \Rightarrow C = 4$$

Hence the solution to the Initial Value Problem (IVP) (an ODE with initial values)

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$$\underbrace{y' = x - e^{x/2}, \quad y(0) = 2}_{\text{IVP}} \quad \text{is} \quad y(x) = \frac{x^2}{2} - 2e^{x/2} + 4.$$

See the red curve above.

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(4) (B) is not of the form $y' = f(x)$.

Rather, it is of the form $y' = f(y)$ ← dependent variable on RHS.

(in this case $p' = f(p)$)

Note: This is harder to solve but easier to study!

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Def if in an ODE the independent variable is <u>NOT</u> explicitly present, the ODE is called <u>autonomous</u>.

Let's solve (6): There are many ways. We choose something like what we will eventually call "separation of variables".

First, recall from <u>Calculus I</u>,

(I) For any diff. $p(t)$, the functions $p(t)$ and $p(t) - 900$ have the same derivative: $p'(t)$.

not directly in text.

(II) $\frac{d}{dt}\left[\ln|f(t)|\right] = \frac{f'(t)}{f(t)}$ for $f(t)$ differentiable and $f(t) \neq 0$.
(chain Rule)

Hence we can rewrite $p' = \frac{p}{2} - 450 = \frac{p-900}{2}$

as $\frac{p'}{p-900} = \frac{1}{2}$. Why is this useful?

It is useful since the LHS of $\frac{p'}{p-900} = \frac{1}{2}$

looks like the derivative

$$\frac{d}{dt}\left[\ln|p(t)-900|\right] = \frac{1}{2}.$$

Integrate both sides as functions of t to get

$$\ln|p(t)-900| = \frac{t}{2} + c$$

and exponentiate to get ~~[illegible]~~

$$|p(t)-900| = e^{t/2+c} = e^{t/2}e^{c}$$ make sense?

If we can "solve" this for p(t) we are done since we would have an expression for the p(t) that solves 6.

Q: Given $|p(t)-900| = e^{t/2}e^{c}$, can $p(t)=900$? Why or why not?

The ODE (I) has 3 types of solutions:

(1) $p(t) > 900$ always $\Rightarrow p(t) - 900 = Ke^{t/2}$

or $p(t) = 900 + Ke^{t/2}$, where $K = e^c > 0$.

(2) $p(t) < 900$ always $\Rightarrow -(p(t) - 900) = Ke^{t/2}$

or $p(t) = 900 + \underbrace{(-K)}_{\text{negative constant}} e^{t/2}$, where $K = e^c > 0$.

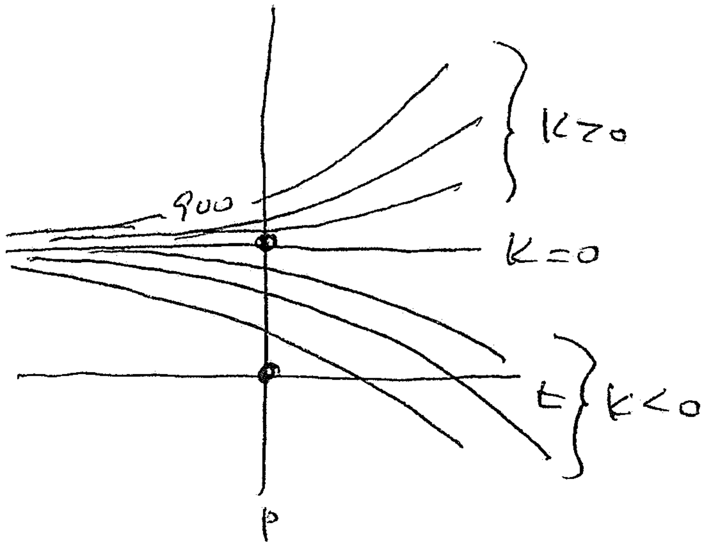
(3) $p(t) = 900$ for all $t \in \mathbb{R}$. It works in the ODE and can also be written

$$p(t) = 900 + Ke^{t/2}, \quad K = 0.$$

This last solution is called a <u>singular solution</u> which sometimes become hidden due to the method employed to find the other solutions.

Note: Our method included a divide by $p - 900$ term which implied we discounted that possibility. We need to account for it.

Conclusion $p(t) = 900 + Ke^{t/2}$, $K \in \mathbb{R}$
is the general solution to (b)



Here the singular solution
$p(t) = 900 \ \forall t$ is
called an equilibrium
solution (or a steady-state soln).

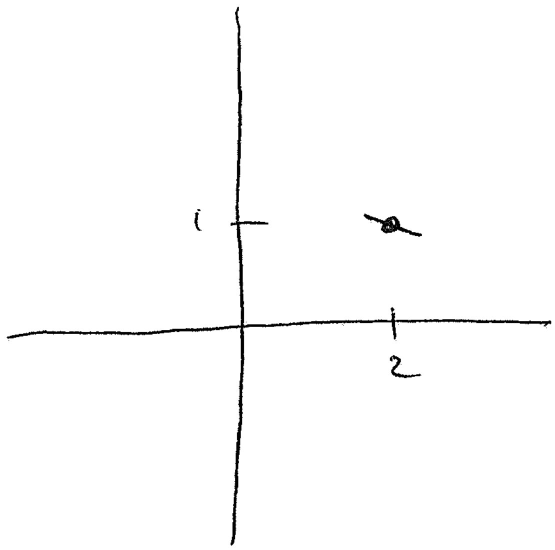
Equilibrium solutions will be very important
to us in the future.

Notice in both (a) and (b) above

$\frac{dy}{dx} =$ something involving x, y $\qquad$ $\frac{dp}{dt} =$ something involving t, p

This is useful since solutions "live" in the xy-plane
or the tp-plane, any solution curve will have
its tangent line at (x,y) with slope given by
simply evaluating the RHS at that point.

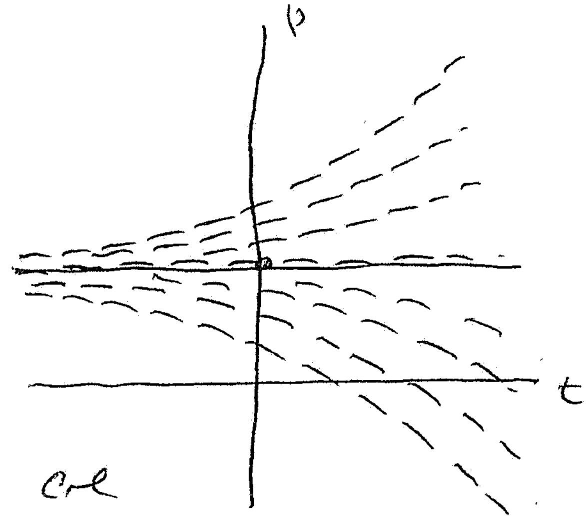
ex. For $y' = x - e^{x/2}$, choose $(x_0, y_0) = (2,1)$.



Then $\left.\dfrac{dy}{dx}\right|_{\substack{x=2\\y=1}} = 2 - e^{2/2} \cong -.718$

The solution curve passing through $(2,1)$ will have slope $\sim -.718$ there.

Without solving the ODE, we can take
pg 3. a grid of pts in the xy-plane and evaluate these little slope lines. The result is called a *slope field*:



like blades of grass in a stream with a strong current.

These line segments are tangent to all solution curves.

We will use these as a method of study over time.