Very generally, a first-order ODE of the form
\[ \frac{dy}{dx} = f(t, y) \]  
will have a solution of both \( t \) and \( y \) and will not be solvable.

However, with some additional structure to \( f \), there are methods to solve. In Chapter 2, we explore some of these.

First type of structure (Section 2.1): Linear

\( f(t, y) = -p(t)y + q(t) \) for some continuous functions \( p(t) \), \( q(t) \).

Then (x) can be rewritten
\[ y' = -p(t)y + q(t) \] or
\( (\forall x) \quad y' + p(t)y = q(t) \)
This new form exposes a structure that facilitates calculation. The LHS is almost the total derivative of a function. To make it so, we multiply the ODE by an expression called an integrating factor.

Def: An integrating factor is a term that when multiplied to an expression renders the expression integrable.

To understand what we are looking for, look at the patterns here:

Let $y$ be a function of $t$. Then, for any other function $f(t)$, we have

$$\frac{d}{dt}[f(t)\cdot y] = f(t)\cdot y' + f'(t)\cdot y \quad \text{by Product Rule}$$
And also \( \frac{d}{dt} [e^{\phi(t)} y] = e^{\phi(t)} y' + e^{\phi(t)} p' \phi(t) y \)

\[ = e^{\phi(t)} [y' + p' \phi(t) y] . \]

We do this just to look for patterns. In this case, we see an important one: inside the bracket, \( [y' + p' \phi(t) y] \) looks very close to the LHS of (2.9) \( y' + p(t) y = q(t) \).

In fact, they are precisely the same when \( p'(t) = p(t) \), or \( \phi(t) = \int p(t) dt \).

So we do one more calculation for a pattern:

\[ \frac{d}{dt} \left[ e^{\int p(t) dt} y \right] = e^{\int p(t) dt} y' + \frac{d}{dt} \left[ e^{\int p(t) dt} \right] y \]

\[ = e^{\int p(t) dt} y' + e^{\int p(t) dt} p(t) y \]

\[ = e^{\int p(t) dt} [y' + p(t) y] . \]

Precisely the LHS of

(2.9) \( y' + p(t) y = q(t) \)
This is useful because, if we take \( y' + p(t)y = qa \), and multiply the entire eqn by \( e^{\int p(t)dt} \), then the LHS becomes easily integrable.

Call \( e^{\int p(t)dt} \) the integrating factor of

\[ y' + p(t)y = q(t). \]

**Challenge Q:** It turns out, any antiderivative of \( p(t) \) will give the same effect. Why?

Let's play this out and see just how the integrating factor is helpful.

Solve \( y' + p(t)y = q(t) \).
Step 1: Multiply each side by $e^{\text{Spt}\text{dt}}$.

$$e^{\text{Spt}\text{dt}} \left[ y' + p(t) y \right] = q(t)$$

$$e^{\text{Spt}\text{dt}} y' + p(t) e^{\text{Spt}\text{dt}} y = e^{\text{Spt}\text{dt}} q(t)$$

$$\frac{d}{dt} \left[ e^{\text{Spt}\text{dt}} y \right] = e^{\text{Spt}\text{dt}} q(t).$$

Step 2: Integrate with respect to (wrt) $t$.

$$\int \frac{d}{dt} \left[ e^{\text{Spt}\text{dt}} y \right] dt = \int e^{\text{Spt}\text{dt}} q(t) dt$$

$$e^{\text{Spt}\text{dt}} y = \int e^{\text{Spt}\text{dt}} q(t) dt + C$$

Step 3: Solve for $y$.

$$y(t) = e^{-\text{Spt}\text{dt}} \left[ \int e^{\text{Spt}\text{dt}} q(t) dt + C \right]$$
Notes

1. Practically, we can always do this. The integrating factor $e^{\int P(t)dt}$ is pretty easy to calculate, usually.

2. You do not need to memorise any thing of the form of step 2. Just remember the steps.

3. Any antiderivative of $P(t)$ will do, since they all only differ by a constant.

You are multiplying the entire equation by the factor.

ex. Suppose $P(t) = 2t$. Then $e^{\int 2t dt} = e^{t^2} = e^t$.

If instead you choose $e^{\int 2t dt} = e^{t^2 + c}$, then

$e^{t^2 + c} = e^{t^2}e^c = e^{t^2}k$, for $K = e^c$ constant.

Then $ke^{t^2}[g'(t) + P(t)g = s(t)]$ is seen as $e^{t^2}[g'(t) + P(t)g = s(t)]$.

As far as solutions are concerned.
Some examples

(1) Solve $ty' - 2y = t^2 e^{-2t}$

**Strategy:** This is linear so we use the int. fact $e^{\int P(t)dt}$ to solve using the 3 steps above.

**Solution:** Place the ODE in standard form

$$y' - \frac{2}{t} y = t^2 e^{-2t}.$$  

This gives us $P(t) = -\frac{2}{t}$, so the int. factor is $e^{\int P(t)dt} = e^{\int -\frac{2}{t} dt} = e^{-2\ln |t|} = e^{\ln t^{-2}} = t^{-2}$.

**Step 1:** Multiply ODE by int. factor.

$$t^{-2} \left[ y' - \frac{2}{t} y = t^2 e^{-2t} \right]$$

$$t^{-2} y' - \frac{2}{t} \cdot \frac{y}{t} = e^{-2t}$$

$$\frac{d}{dt} \left[ t^{-2} y \right] = e^{-2t}$$

**Step 2:** Integrate w.r.t. $t$.

$$\int \frac{d}{dt} \left[ t^{-2} y \right] dt = t^{-2} y + c_1 = \int e^{-2t} dt = -\frac{1}{2} e^{-2t} + c_2$$

$$t^{-2} y = -\frac{1}{2} e^{-2t} + K$$

**Step 3:** Solve for $y(t)$:

$$y(t) = -\frac{1}{2} t^2 e^{-2t} + K t^2$$

This $y(t)$ solves the ODE.
Check to see if this is correct:

\[
\begin{align*}
-t\,e^{-2t} + t^2\,e^{-2t} + 2kt &- \frac{2\,k}{t} \left( -\frac{1}{2} t^2 e^{-2t} + kt^2 \right) = t^2 e^{-2t} \\
\frac{d}{dt} \left[ t^2 e^{-2t} \right] &- 2te^{-2t} + 2kt + t^2 e^{-2t} - 2kt^2 = t^2 e^{-2t} \\
t^2 e^{-2t} &= t^2 e^{-2t} 
\end{align*}
\]

Solution:

\( x + 2tx = t^3 \). Solve this.

**Strategy:** Use the integrating factor on this linear ODE to integrate through to an expression for \( x(t) \).

**Solution:** This ODE is linear, with \( p(t) = 2t \).

Thus the int. factor is

\[ e \int_{\text{left}} \, dt = e^{\int 2t \, dt} = e^{t^2} \]

**Step 1:** Multiply ODE by int. factor:

\[ e^{t^2} \left[ x + 2tx = t^3 \right] \]

\[ e^{t^2} x + 2te^{t^2} x = t^3 e^{t^2} \]

\[ \frac{d}{dt} \left[ e^{t^2} x \right] = t^3 e^{t^2} \]
Step 2: Integrate with t.

\[ \int \frac{d}{dt} [e^t \cdot x] \, dt = e^t \cdot x + C_1 = \int e^t \cdot t^2 \, dt \]

\[ \int e^t \cdot t^2 \, dt = \frac{\text{Subst.}}{s = t^2} \frac{1}{2} \int s \cdot e^s \, ds \]

\[ \frac{1}{2} \int s \cdot e^s \, ds = \frac{1}{2} (s \cdot e^s - \int e^s \, ds) = \frac{1}{2} (s \cdot e^s - e^s) + C_2 \]

Combine constants to get:

\[ e^t \cdot x = \frac{1}{2} e^{t^2} (t^2 - 1) + K \]

Step 3: Solve for \( x(t) \).

**This is the general solution to** \( x + 2tx = t^3 \)

\[ x(t) = \frac{1}{2} t^2 - \frac{t}{2} + Ke^{-t^2} \]

Check this:

\[ \left( t - 2Kt \cdot e^{-t^2} \right) + 2t \left( \frac{1}{2} t^2 - \frac{t}{2} + Ke^{-t^2} \right) = t^3 \]

\[ t^2 - 2Kt \cdot e^{-t^2} + t^3 - t + 2Kt \cdot e^{-t^2} = t^3 \]

\[ t^3 = t^3 \quad \text{of works} \]
Solve \( \frac{dx}{ds} = \frac{x}{s} - s^2 \), for \( s > 0 \)

Here the ODE is again linear (note \( s \) is the independent variable), and \( p(s) = -\frac{1}{s} \).

The integrating factor is then
\[
e^{\int p(s)ds} = e^{\int -\frac{1}{s} ds} = e^{-\ln s} = e^{-\ln s} s^{-1} = s^{-1}
\]

Multiply through standard form of ODE to get

\[
\frac{1}{s} \left[ \frac{dx}{ds} - \frac{x}{s} = -s^2 \right] \Rightarrow \frac{1}{s} \frac{dx}{ds} - \frac{x}{s^2} = -s
\]

Integrate wrt \( s \) to get

\[
\frac{1}{s} x = \int (-s) ds + C = -\frac{s^2}{2} + C
\]

Solve for \( x(s) \):

\[
x(s) = -\frac{s^3}{2} + Cs
\]

This is the general solution to ODE

Check it:

\[
\left( -\frac{3}{2} s^2 + C \right) = \frac{1}{s} \left( -\frac{s^3}{2} + Cs \right) - s^2
\]

\[
\frac{\frac{d}{ds}}{x} \left( -\frac{3}{2} s^2 + C \right) = -s^2
\]

\[
-\frac{3}{2} s^2 + C = -\frac{s^2}{2} + C - s^2
\]

\[
-\frac{3}{2} s^2 = -\frac{3}{2} s^2 \quad \sqrt{\text{It is correct.}}
\]
Find the general solution to
\[
 t(y' - y) = (1 + t^2) e^t \quad \text{on } t > 0.
\]

Here, try to see why this is linear, with
\[
p(t) = -1.
\]

The solution is
\[
y(t) = e^t (\ln t + \frac{t^2}{2} + c)
\]

This solution is drawn up in a separate document under example problems on the website.

Solve \( \frac{dp}{dt} = \frac{p}{2} - 450 \) using an integrating factor.

Solution: This is an exercise. You already know the answer.