110.302 Lecture 4: [redacted]

(II) Structure type: Seperable

Suppose for $y' = f(t,y)$ that

$$f(t,y) = g(t)\,h(y)$$

for 2 functions $g(t)$, and $h(y)$.

different way from book.

Then we say the ODE is <u>seperable</u>

(we can seperate RHS into the product of 2 functions; one of t alone and the other of y alone).

Then the ODE is $y' = g(t)\,h(y)$, and we can write

$$\frac{1}{h(y)}\frac{dy}{dt} = g(t)$$

Since y is a function of t, both sides are fncs of t and we can integrate <u>wrt</u> t.

$$(\star) \quad \underbrace{\int \left(\frac{1}{h(y)}\frac{dy}{dt}\right)dt}_{\substack{\text{the antiderivative} \\ \text{of } \frac{1}{h(y)} \text{ as a} \\ \text{function of } t}} = \underbrace{\int g(t)\,dt}_{\substack{\text{the antiderivative} \\ \text{of } g(t)}}$$

The general solution to this kind of ODE is then found by integration alone.

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ex. Find the general solution to $\frac{dy}{dx} = xy^2$

(Note: This ODE is separable, but NOT linear!)

Solution: Separate the variables: $\frac{1}{y^2}\frac{dy}{dx} = x$

Then integrate both sides wrt x:

$$\int \frac{1}{y^2}\frac{dy}{dx}\,dx = \int x\,dx$$

$$-\frac{1}{y} = \frac{x^2}{2} + C = \frac{x^2+K}{2}$$

$$\Longrightarrow \boxed{y = \frac{-2}{x^2+K}}$$

Note: While this is the general soln, particular solutions will require more than just a choice of K!

This function works since for $y = \frac{-2}{x^2+k}$

$$y' = \frac{2}{(x^2+k)^2} \cdot 2x = x\left(\frac{2}{x^2+k}\right)^2 = x\left(\frac{-2}{x^2+k}\right)^2 = xy^2 \quad \checkmark.$$

Notes ① The LHS of (✱) is interesting:

$\int \left(\frac{1}{h(y)} \frac{dy}{dt}\right) dt$ is the antiderivative of $\frac{1}{h(y)}$ as a function of t; in the example,

$$\int \frac{1}{y^2} \frac{dy}{dx}\, dx = \frac{-1}{y} + c.$$

To see this, rewrite LHS(✱) using u as the dependent variable:

$\int \left(\frac{1}{h(u(x))} \cdot \frac{du}{dx}\right) dx$ looks just like the integrand one would find in a substitution problem:

Let $y = u(x)$, $dy = u'(x)\, dx = \frac{du}{dx}\, dx$

Then $\int \frac{1}{h(u(x))} \frac{du}{dx}\, dx = \int \frac{1}{h(y)}\, dy$ and you can integrate wrt y directly!

In our example above,

$$\int \frac{1}{y^2} \frac{dy}{dx}\, dx = \int \frac{1}{y^2}\, dy = \frac{-1}{y} + c.$$

Strictly speaking one does NOT simply cross out the dx's. But it does look that way.

(2) The book uses a slightly different formula:

Any $y' = \frac{dy}{dx} = f(x,y)$ can be written

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

(think $M = -f$ and $N = 1$, but this may sometimes not be the only way).

Then if $M(x,y) = M(x)$ and $N(x,y) = N(y)$

we get $M(x) + N(y)\frac{dy}{dx} = 0$ or

$N(y)\frac{dy}{dx} = M(x)$ and ODE is separable.

(3) In differential form

$$M(x) + N(y)\frac{dy}{dx} = 0$$

may be presented $M(x)\,dx + N(y)\,dy = 0$

or $\quad N(y)\,dy = -M(x)\,dx.$

Integrating the differentials yields

$$-\int M(x)\,dx = \int N(y)\,dy$$

ex. We may sometimes see $y' = xy^2$ as

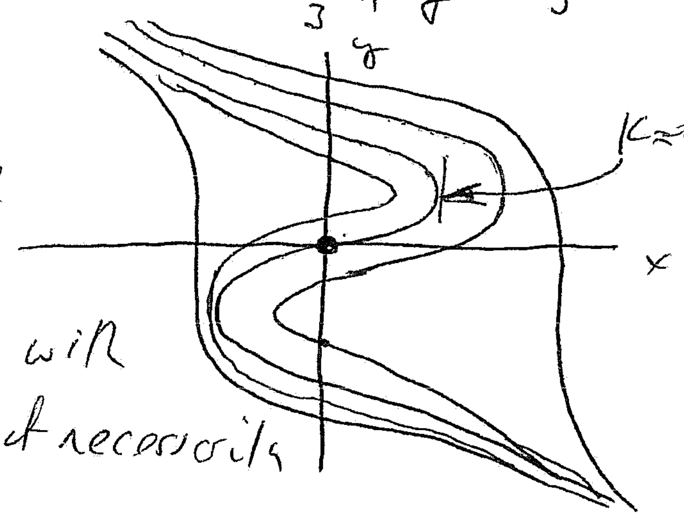
$\frac{dy}{dx} = xy^2$, or $\frac{dy}{dx} - xy^2 = 0$, or $\frac{dy}{y^2} = x\,dx$

The notation is different, but the ODE is the same.

(4) Sometimes, a solution is known only implicitly:

<u>ex in book:</u> $\frac{dy}{dx} = \frac{x^2}{1-y^2}$

Solution is $-\frac{x^3}{3} + y - \frac{y^3}{3} = K$



These curves are solutions to a general eqn with x and y. (not necessarily a function).

K=0 level set of $H(x,y) = \frac{-x^3}{3} + y - \frac{y^3}{3}$

Q: How does one use this information to find an explicit solution?

Q: What is the domain of each solution?

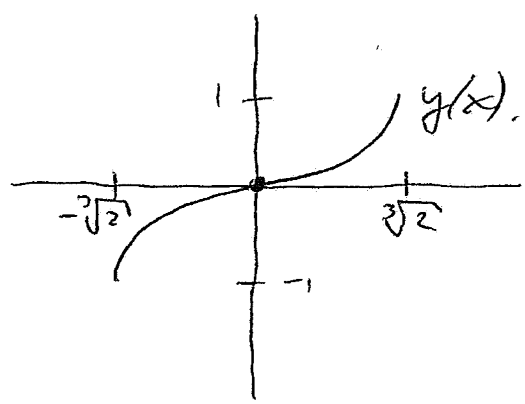
Q: How do we know which piece to pick?

For $y' = \frac{x^2}{1-y^2}$, $y(0)=0$, the solution is $y(x)$ where $-\frac{x^3}{3}+y-\frac{y^3}{3}=0$, but the function $y(x)$ is only defined up to the vertical tangent lines: these are where $y^2=1$, or $y=\pm 1$.

Here when $y=1$, $-\frac{x^3}{3}+1-\frac{1}{3} \Rightarrow x=\sqrt[3]{2}$.

$y=-1 \Rightarrow x=-\sqrt[3]{2}$.

Caution: A solution to an ODE is a function (even when defined implicitly) that <u>includes</u> its <u>domain</u>!



Solution is integral curve of $-\frac{x^3}{3}+y-\frac{y^3}{3}=0$, on interval $-\sqrt[3]{2} < x < \sqrt[3]{2}$, that incl. $(0,0)$.

⑤ Back to $y' = xy^2$ with its general solution $y(x) = \frac{-2}{x^2+K}$. This is fine as a general solution.

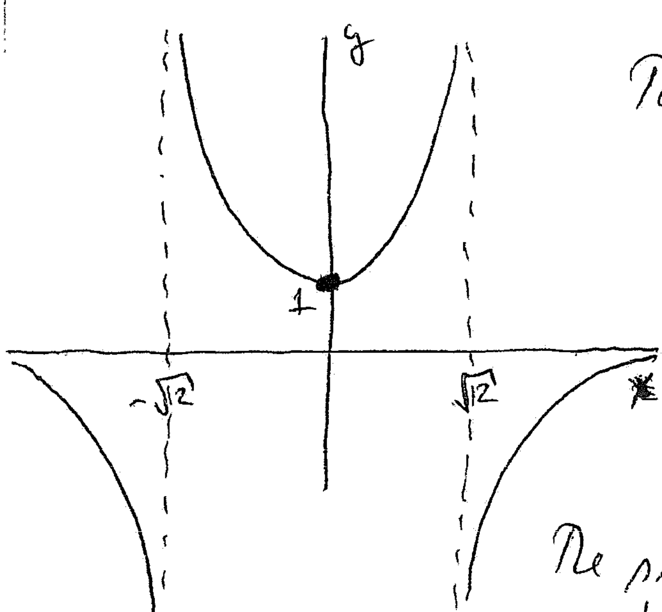
But for an IVP, we will also need a domain for which the solution is continuous

IVP: $y' = xy^2$, $y(0) = 1$.

Here, the particular solution has K-value $K = -2$

But $y(x) = \frac{-2}{x^2-2}$ does not solve $y' = xy^2$, $y(0) = 1$.

Only the <u>continuous piece</u> that contains the initial value <u>is</u> the solution.



The solution to $y' = xy^2$, $y(0) = 1$ is $y(x) = \frac{-2}{x^2-2}$ on $(-\sqrt{2}, \sqrt{2})$ only.

The solution to $y' = xy^2$, $y(2) = -1$ is $y(x) = \frac{-2}{x^2-2}$ on $(\sqrt{2}, \infty)$ only.

The proper domain is absolutely necessary to specifying a solution.

One more example.

example (ex 4 pg 37)

Solve the IVP $ty' + 2y = 4t^2$ for

ⓐ $y(-1) = 1$, ⓑ $y(-1) = 2$, ⓒ $y(0) = 0$, ⓓ $y(0) = 1$.

Here method of int. factors yields

$y(t) = t^2 + \frac{c}{t^2}$ as the general soln.

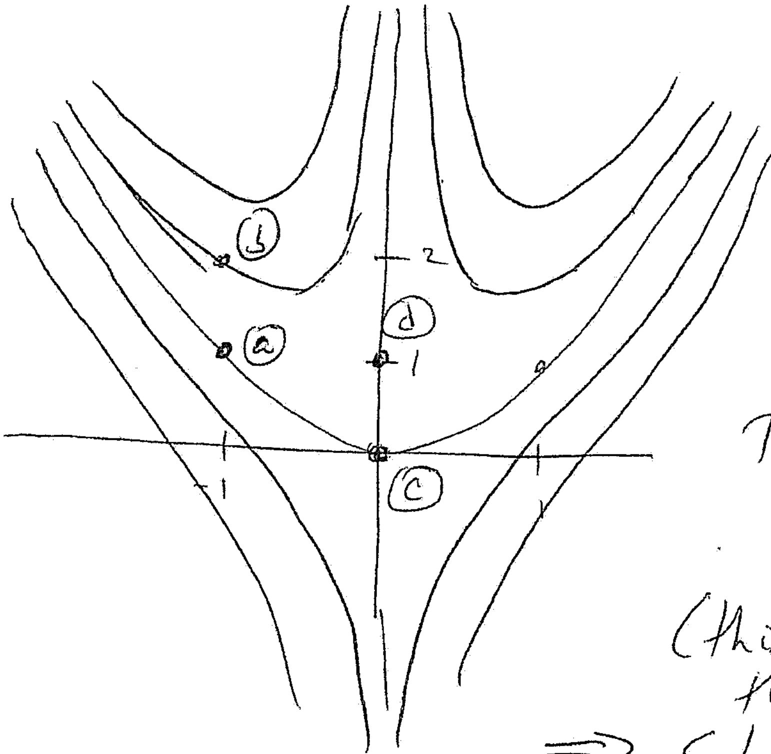
ⓐ $y(-1) = 1 = (-1)^2 + \frac{c}{(-1)^2} \Rightarrow c = 0$ $y(t) = t^2$ for $t \in (-\infty, \infty)$.

ⓑ $y(-1) = 2 = (-1)^2 + \frac{c}{(-1)^2} \Rightarrow c = 1$ $y(t) = t^2 + \frac{1}{t^2}$ for $t \in (-\infty, 0)$

ⓒ Cannot plug in 0. But the pt $(0,0)$ is on an integral curve of IVP. It is on the curve $y = t^2$ on $(-\infty, \infty)$.

ⓓ The pt $t=0, y=1$ is not on any integral curve. The IVP $ty' + 2y = 4t^2$, $y(0) = 1$ has no solution. What gives??

IX



The domain of the IVP solution in ⓐ is $(-\infty, \infty)$

The domain of IVP in ⓑ is $(-\infty, 0)$

(this is only 1 piece of $y(t) = t^2 + \frac{1}{t^2}$ the one that includes the pt.

$\Rightarrow$ Soln to IVP $ty' + 2y = 4t^2$, $y(\cancel{-1}) = 1$ is

$$\boxed{y(t) = t^2 + \frac{1}{t^2} \text{ on } (-\infty, 0)}$$

Careful here.