110.302 Lecture 6,7 I

New Structure type: Autonomous

Suppose in $y' = f(t, y)$, f is only a function of y: $y' = f(y)$

Such an ODE IS separable, but $\frac{1}{f(y)} \frac{dy}{dt} = 1$ may still be hard to integrate.

An ODE of the form $y' = f(y)$ is called autonomous: t is not explicitly present in the equation.

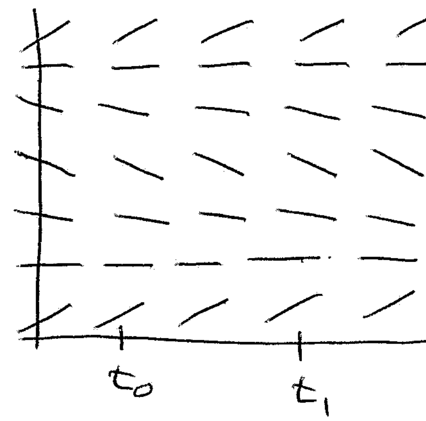
exs $\dot{x} = x^{4/3}$, $y' = ky$, k a constant

$\frac{dz}{dt} = z(1-z)$ (Logistic eqn).

Here, even without solving, properties of autonomous ODEs allow for effective study.

Properties of autonomous $y' = f(y)$

① <u>Structure of slope field.</u>



- Slope field doesn't change in t-direction. (different t's have same look)

  – Every vertical slice <u>looks the same</u>.

- Every horizontal slice is an <u>isocline</u>: a curve along which all slopes of solution curves are the same.

② <u>Existence and uniqueness:</u>

Since here f is only a function of y,

- Existence of solutions is assured when $f(y)$ is continuous.

- Uniqueness of solutions is assured when $f'(y) = \frac{df}{dy}$ is continuous.

(Here, $\frac{\partial f}{\partial y}(t,y) = \frac{df}{dy} = f'(y)$ since t does not occur as a variable in f).

Conclusion: No crossings of solutions where $f(y)$ and $f'(y)$ are defined.

③ Equilibrium Solutions

At any place y_0 where $f(y_0) = 0$, then $y'(t) = 0$ here, and this $y(t) = y_0$ is a constant solution (or equilibrium, or steady-state solution).

Its graph is a horizontal line and is an isocline.

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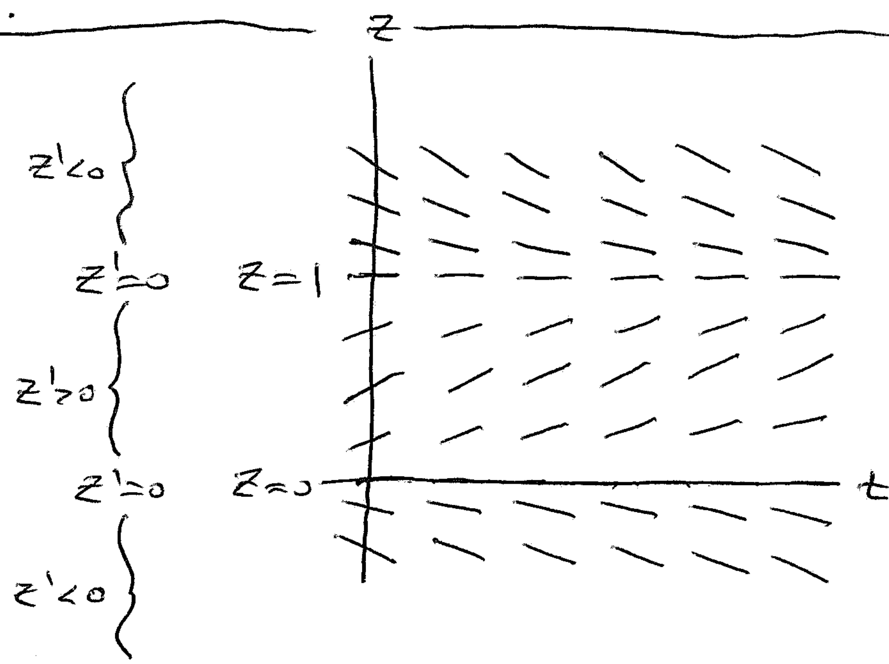
ex $\quad z' = z(1-z)$

Here $z(t) = 0$ and $z(t) = 1$ are both equilibrium solutions



And in between the equilibria, the sign of z' does not change. Hence solutions

Ⓐ are trapped between equilibria, and

Ⓑ always travel in the same direction.

In the example, we can say the following without solving:

(I) Solutions exist and are unique everywhere ($f(z)$ and $f'(z)$ are polynomials)

(II) Equilibria only at $z=0$ and $z=1$.

(III) Any solution that passes through $0<z_0<1$ will tend toward the equilibria $z(t)=1$.

Any solution that starts at $z_0<0$ will tend to $-\infty$

Any solution that starts at $z_0>1$ will tend to $z(t)=1$

Here we can say
$$\lim_{t\to\infty} z(t) = \begin{cases} 1 & z_0>0 \\ 0 & z_0=0 \\ -\infty & z_0<0 \end{cases}$$

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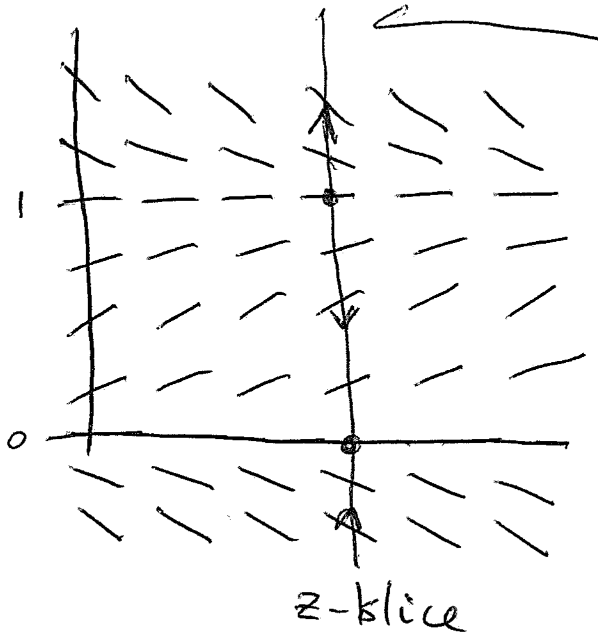
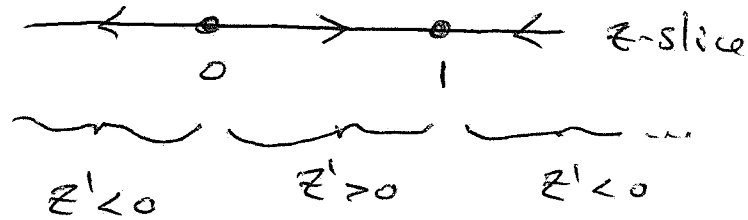
(4) Phase line

Any vertical slice through the slope field gives you all information about long term behavior of solutions:



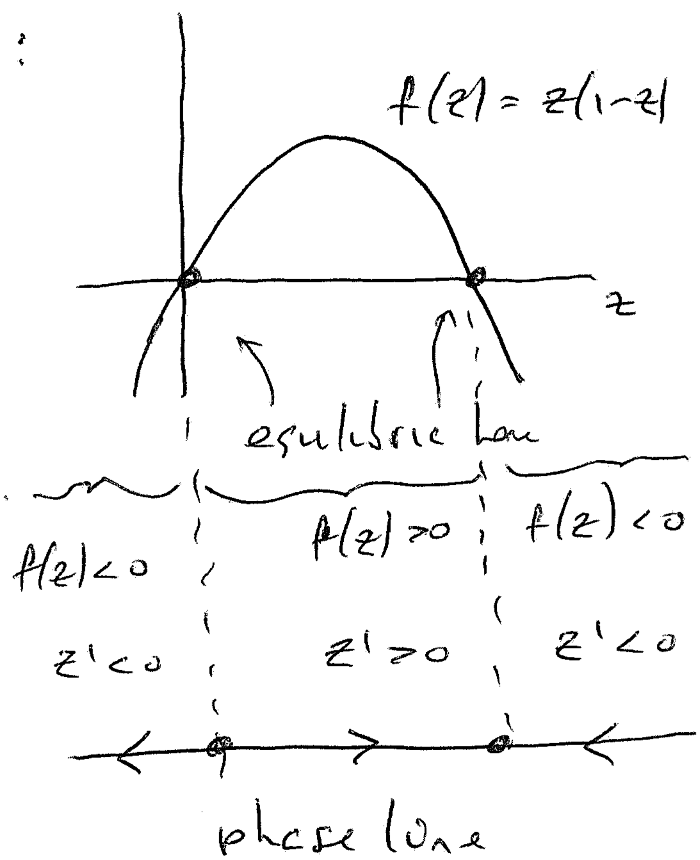
Phase line of $z' = z(1-z)$



Here, the phase line is a schematic that determines all long term behavior of the autonomous $z' = f(z)$

without a slope field, still easy to see phase line: Graph $f(z)$:

$$z' = z(1-z) = f(z)$$



Def. For $y' = f(y)$, the set $\{y \in \mathbb{R} \mid f(y) = 0\}$ is the set of <u>critical pts</u> for the ODE. (equilibrium solutions)

Critical pts (equilibrium solutions) can be classified by how solutions behave around them:

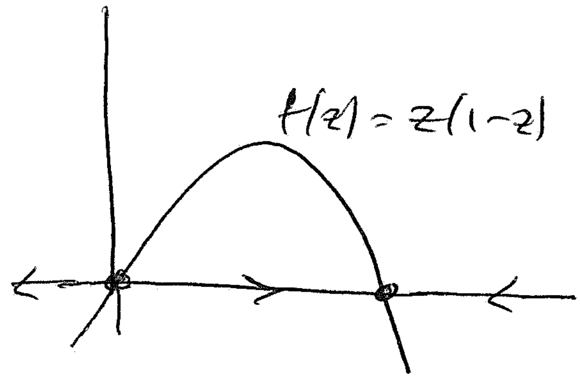
Let y_* be a critical pt for $y' = f(y)$, and let $N_\varepsilon(y_*) = \{ y \in \mathbb{R} \mid |y - y_*| < \varepsilon \}$ be an ε-neighborhood of y_*,

ⓐ If there is an $\varepsilon > 0$ where for all $y \in N_\varepsilon(y_*)$ $\lim_{t \to \infty} y(t) = y_*$ $\Rightarrow$ y_* is <u>asymptotically stable</u>

ⓑ If there is an $\varepsilon > 0$ where for all $y \in N_\varepsilon(y_*)$ $\lim_{t \to -\infty} y(t) = y_*$ $\Rightarrow$ y_* is unstable

ⓒ If asympt. stable on one side and unstable on the other, then y_* is semi-stable.

ex. $z' = z(1-z)$. Here critical pts are $z = 0, 1$. And



Here $z(t) = 1$ is asymptotically stable and $z(t) = 0$ is unstable.

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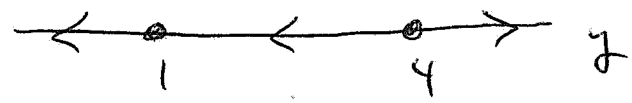
ex. $y' = (1-y)^2(y-4)$

Here, critical pts at $y = 1, 4$.

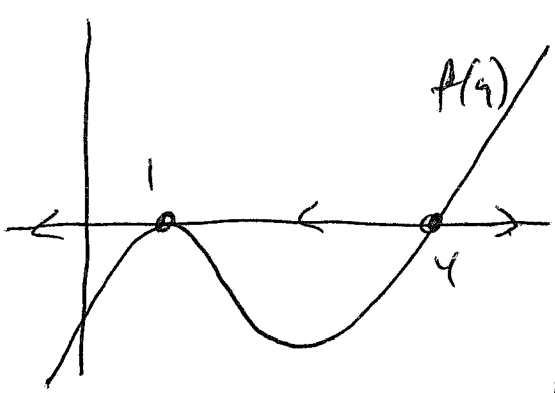
Phase line is

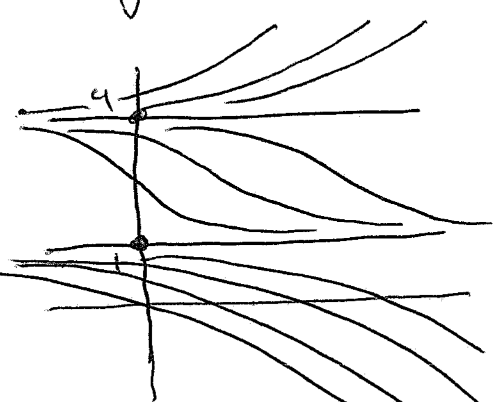


(check a pt in each interval formed by critical pts.)

$y(t) = 4$ is unstable
$y(t) = 1$ is semistable.

Graph of ~~g~~ $f(y) = (1-y)^2(y-4)$





$$\lim_{t\to\infty} y(t) = \begin{cases} \infty & y_0 > 4 \\ 4 & y_0 = 4 \\ 1 & 1 \le y_0 < 4 \\ -\infty & y_0 < 1 \end{cases}$$

When graphing the $f(y)$ in $y' = f(y)$ and in constructing phase lines, some patterns develop:

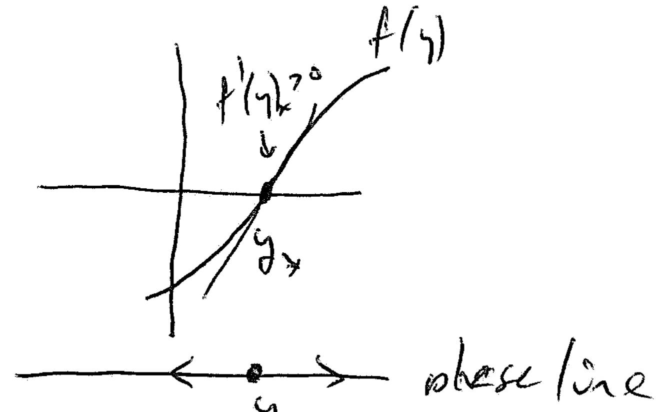
Let y_* be an equilibrium for $y' = f(y)$ (thus $f(y_*) = 0$).

For y "near" y_*, $y' = f(y) \cong \overbrace{\cancelto{0}{f(y_*)} + f'(y_*)(y - y_*)}^{\text{1st Taylor approx to } f(y) \text{ at } y_*}$

Case: Suppose $f'(y_*) > 0$

$\Longrightarrow$ for $y > y_*$, $y' > 0$

$y < y_*$, $y' < 0$



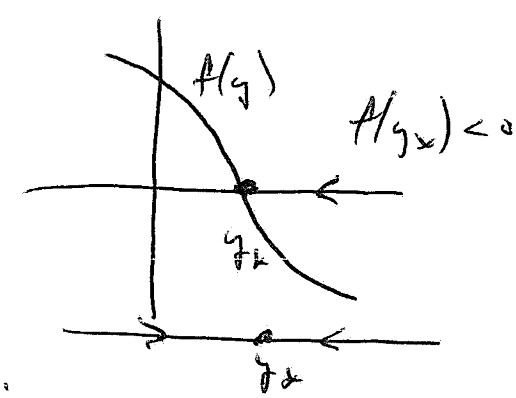
All nearby solutions move away

$\Longrightarrow$ y_* is an unstable node or source

Case 2: for $f'(y_*) < 0$

$\Longrightarrow$ for $y > y_*$ $y' < 0$

$y < y_*$ $y' > 0$



All nearby soln converge to y_*.

$\Longrightarrow$ asymptotically stable or sink.

Case 3: $f'(y_*) = 0$. Need more information.