110.302 Lecture 8

Bifurcations

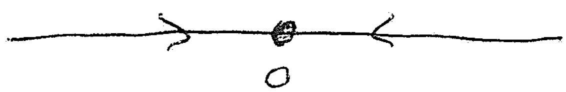
Consider the autonomous $y' = f(a,y)$ where "a" is a parameter (an unknown constant)

The number and classification of equilibria may depend on the value of "a"

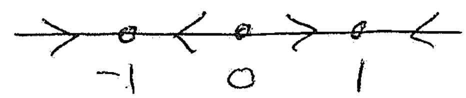
ex. $\dot{y} = ay - y^3 = y(a - y^2)$

Here, for $a<0$ and $a>0$ the number of equilibria are different (also the type)

$a = -1 < 0$



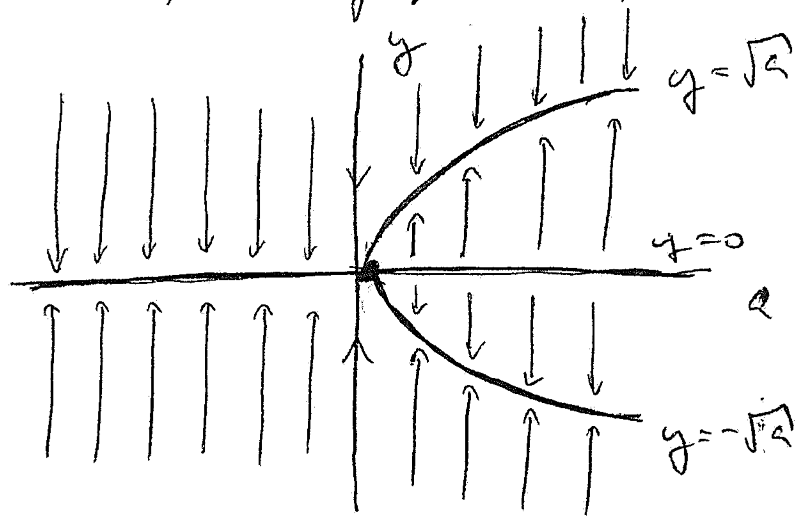
$a = 1 > 0$



Here $f(-1,y) = -y(1+y^2)$ and $y(t) \equiv 0$ is the only equilibrium. It is asympt. stable

Here $f(1,y) = y(1-y^2)$
$= y(1-y)(1+y)$
and equilibria exist at $y = -1, 0, 1$. $y(t) \equiv 0$ is unstable here.

We can study how the parameter affects equilibria via a <u>bifurcation</u> <u>diagram</u>: a graph of equilibria in relation to parameter value in the ay-plane for $y' = f(a,y)$.



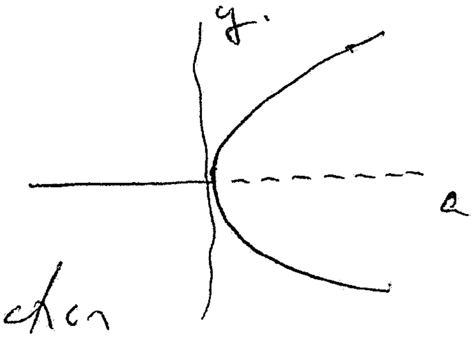
<u>Properties</u>

- each vertical slice here <u>is</u> the phase line of $y' = f(a,y)$ for that value of "a".
- As a varies, equilibria trace out curves of fixed pts. Found by solving $f(a,y) = 0$.
- Special values of "a" where the number of equilibria and/or the stability change are called bifurcation <u>values</u> of "a".
- Here, curves for $a > 0$ correspond to solutions to $f(a,y) = y(a - y^2) = 0$, or to $\boxed{y = 0}$ and $a = y^2$ or $\boxed{y = \pm\sqrt{a}}$

Q: For $a = 9$, find $\lim_{t\to\infty} y(t)$ when $y(0) = e$

- Here, the only bifurcation value of a is $a=0$.
- Here we use solid lines for all equilibria, and vertical arrows to denote stability.

Book uses solid for asympt. stable curves and dotted for unstable curves.



- This kind of bifurcation is called a <u>pitchfork bifurcation</u>

<u>why?</u>

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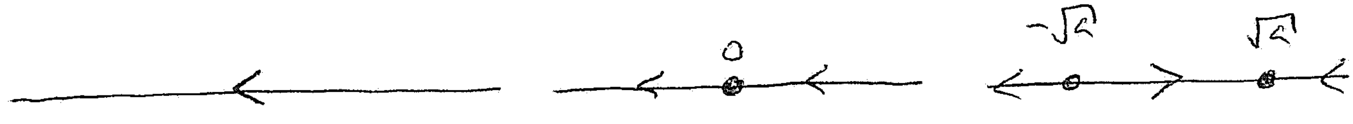
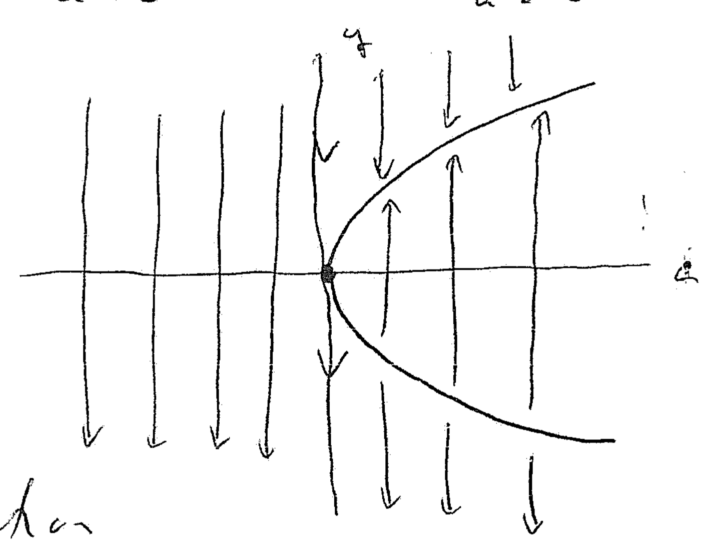
ex. $\dot{y} = a - y^2$



Lines of equilibria solve $a = y^2$ or $y = \sqrt{a}$, $y = -\sqrt{a}$ only for $a \geq 0$

Called a <u>saddle-node</u> or <u>creation</u> bifurcation



ex. Basic model of a laser (simple)

$$\dot{n} = (GN_0 - k)n - Gn^2$$

models a basic simple laser, where

$n(t)$ = # of photons at time t

N_0, k, G are positive constants

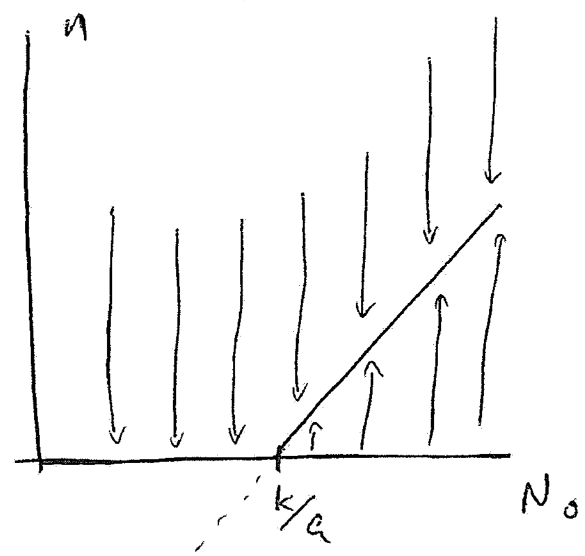
We study how the eqn is affected by $N_0 \geq 0$

Here equilibria are at

$$n(GN_0 - k - Gn) = 0$$

namely $n = 0$

$n = N_0 - k/G$



(I) when $N_0 < k/G$,

$(GN_0 - k) < 0$,

so $\dot{n} < 0$.

$n(t) \equiv 0$ is a sink.

(II) When $N_0 > k/G$, $GN_0 - k > 0 \Rightarrow$ for small n,

$GN_0 - k - Gn > 0$, or $N_0 - \frac{k}{G} - n > 0$, so

for small n, $\dot{n} > 0$. etc.

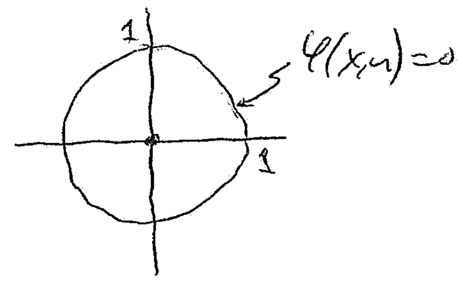
(III) Called a transcritical bifurcation

Change tack

Any equation involving x, y

- indicates that y is an implicit function of x.
- by bringing all terms to LHS may be viewed as the 0-level set of a function of x, y:

ex. $y^2 = 1 - x^2$. Here y is an implicit fnc of x but by creating $\varphi(x,y) = y^2 + x^2 - 1$, the graph of $y^2 = 1 - x^2$ is also the 0-level set of $\varphi(x,y)$: the set of all pts $(x,y) \in \mathbb{R}^2$ that have function value $0 \in \mathbb{R}$.



Either by implicit differentiation on $y^2 = 1 - x^2$, or by differentiation of $\varphi(x,y) = 0$, we can find the slopes of tangent lines to the graph of $y^2 = 1 - x^2$ in $\mathbb{R}^2$.

ex. $y^2 = 1 - x^2$ $\qquad \frac{d}{dx}\left[y^2 = 1 - x^2\right]$

$$2y\frac{dy}{dx} = 0 - 2x \Rightarrow \frac{dy}{dx} = \frac{-x}{y}.$$

$$\varphi(x,y) = y^2 + x^2 - 1 = 0 \qquad \frac{d}{dx}\left[\varphi(x,y)\right] = \frac{d\varphi}{dx} + \frac{d\varphi}{dy}\cdot\frac{dy}{dx}$$

$$= 2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x}{y}.$$

Notice the form of the last calculation. It is written in the language of vector calculus but is a single var. calculus calc.

$\varphi(x,y)$ is a function of 2 vars. But its level sets force y to be an implicit fnc of x.

Hence $\frac{d}{dx}\left[\varphi(x,y) = 0\right] \Rightarrow \boxed{\frac{d\varphi}{dx}(x, y(x)) = 0 = \frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial y}\frac{dy}{dx}}$ (*)