110.302 Lecture 9

Given any 1^{st} order ODE in the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0, \qquad (*)$$

Suppose we could find a function $\varphi(x,y)$ where

① $\frac{\partial \varphi}{\partial x}(x,y) = M(x,y)$

② $\frac{\partial \varphi}{\partial y}(x,y) = N(x,y)$

Then

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \qquad (*)$$

$$\parallel$$

$$\underbrace{\frac{\partial \varphi}{\partial x}(x,y) + \frac{\partial \varphi}{\partial y}(x,y)\frac{dy}{dx}}_{\frac{d}{dx}\left[\varphi(x,y(x))\right]} = 0 \qquad (*)$$

$$\frac{d}{dx}\left[\varphi(x,y(x))\right] = 0 \qquad (+)$$

And solving the ODE $(*)$ is the same as integrating $(+)$ with respect to x, yielding

$$\varphi(x,y) = C$$

as the general (implicit) solution to $(*)$.

ex 1 pg 95 in book

Given the ODE $2x + y^2 + 2xy\,y' = 0$,

we have the form (*) when $M(x,y) = 2x + y^2$

$N(x,y) = 2xy$.

If we let $\varphi(x,y) = x^2 + xy^2$, then

$$\frac{\partial \varphi}{\partial x}(x,y) = 2x + y^2 = M(x,y), \quad \frac{\partial \varphi}{\partial y}(x,y) = 2xy = N(x,y)$$

So

$$2x + y^2 + 2xy\,y' = 0$$

$$\underbrace{\frac{\partial \varphi}{\partial x}(x,y) + \frac{\partial \varphi}{\partial y}(x,y)\frac{dy}{dx}} = 0$$

$$\frac{d}{dx}\left[\varphi(x,y)\right] = 0$$

and integration wrt x yields

$$\varphi(x,y) = x^2 + xy^2 = C$$

as the general solution to the ODE.

extra note:

Solve the IVP $2x + y^2 + 2xyy' = 0$, $y(1) = -2$.

Here the general solution is $x^2 + xy^2 = c$,

so for $x=1$, $y=-2$, and $(1)^2+(1)(-2)^2 = 5$

and the implicit particular solution is

$$x^2 + xy^2 = 5, \quad \text{or} \quad y = \pm\sqrt{\frac{5-x^2}{x}}$$

and we choose the negative branch since $y(1) = -2$

and note the domain is $(0, \sqrt{5}]$, so

$$y(x) = -\sqrt{\frac{5-x^2}{x}} \quad \text{on } (0, \sqrt{5}] \text{ solves the IVP.}$$

---

2 Questions:

Q1: How do we know when such a $\varphi(x,y)$ exists?

Q2: How to we find it when we do know?

Calculus III Theorem (Schwarz, Clairaut).

If $\varphi(x,y)$ has continuous partials in some open region in the plane, then

$$\frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial x}\right)$$

i.e., mixed 2nd partials are equal.

How to apply this to ODEs?

Given $M(x,y) + N(x,y)\frac{dy}{dx} = 0$, then if such a $\varphi(x,y)$ exists, where $\frac{\partial \varphi}{\partial x} = M$ and $\frac{\partial \varphi}{\partial y} = N$, then the mixed partials criterion is

$$\frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial x}\right) = \underbrace{\frac{\partial}{\partial y}(M) = \frac{\partial}{\partial x}(N)} = \frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial y}\right)$$

also written $\boxed{M_y = N_x}$

Let's put this together into something useful.

Def The ODE $M(x,y) + N(x,y)\frac{dy}{dx} = 0$
is called **exact** on a region
$$R = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{matrix} \alpha < x < \beta \\ \gamma < y < \delta \end{matrix} \right\}.$$
if ① M, N, M_y, N_x are continuous on $\mathbb{R}$,
② $M_y = N_x$ on $\mathbb{R}$.

Thm If $M(x,y) + N(x,y)\,y' = 0$ is exact on a region R, then there is a function $\psi(x,y)$ which is differentiable on R, where

① $\frac{\partial \psi}{\partial x} = M$, ② $\frac{\partial \psi}{\partial y} = N$, and

$\psi(x,y) = c$ is the general solution to the ODE on R.

In practice?

ex. Solve $(3x^2-2xy+2)+(6y^2-x^2+3)y'=0$

Strategy: First, we verify exactness. Then we integrate to find the function whose level sets comprise solutions to the ODE.

Solution: Here
$$M(x,y)=3x^2-2xy+2$$
$$N(x,y)=6y^2-x^2+3$$

and since $M_y=\frac{\partial M}{\partial y}=-2x=\frac{\partial N}{\partial x}=N_x$,

the ODE is exact.

And since M, N, M_y, N_x are all cont on $\mathbb{R}^2$ (they are all polynomials in x,y), by Thm $\varphi(x,y)$ exists on $\mathbb{R}^2$.

To find $\varphi(x,y)$, integrate $\frac{\partial \varphi}{\partial x}=M$ wrt x:

$$\int M\,dx=\int \frac{\partial \varphi}{\partial x}\,dx=\int(3x^2-2xy+2)\,dx=$$
$$=x^3-x^2y+2x+h(y)$$ why?

Hence $\varphi(x,y) = x^3 - x^2y + 2x + h(y)$
for some unknown function $h(y)$.

To find $h(y)$, note that $\frac{\partial \varphi}{\partial y} = N$, so

$$\frac{\partial}{\partial y}[\varphi(x,y)] = \frac{\partial}{\partial y}[x^3 - x^2y + 2x + h(y)].$$
$$= -x^2 + h'(y)$$
$$= N(x,y) = 6y^2 - x^2 + 3$$

Hence $h'(y) = 6y^2 + 3$, so $h(y) = 2y^3 + 3y$ + const.

Thus our general (implicit) solution is

$$\boxed{\varphi(x,y) = x^3 - x^2y + 2x + 3y + 2y^3 = C}$$

Now try this on our first example:

$$2x + y^2 + 2xyy' = 0, \quad y(1) = -2.$$

Last notes

① Sometimes, the ODE is written as a differential:

ex. $(ye^{2xy} + x)\,dx + xe^{2xy}\,dy = 0$

is exact, since $M(x,y) = ye^{2xy} + x$

$N(x,y) = xe^{2xy}$

and $\left.\begin{aligned} M_y &= e^{2xy} + 2xye^{2xy} \\ N_x &= e^{2xy} + 2xye^{2xy} \end{aligned}\right\}$ equal so exact.

The term "exact" comes from this interpretation of ~~[illegible]~~ the LHS of this ODE as an exact differential one-form.

---

② Sometimes, a non-exact 1st order ODE can be made exact via a integration factor:

ex. $dx + \left(\frac{x}{y} - \sin y\right) dy = 0$ is not exact,

since $M_y = 0 \neq \frac{1}{y} = N_x$.

But $y\left[dx + \left(\frac{x}{y} - \sin y\right) dy = 0\right]$

$$y\,dx + (x - y\sin y)\,dy = 0$$

is exact, since now $M_y = 1 = N_x$.

We will not focus on this technique, though.