110.302 Lecture 10

I will relegate the discussion of Section 2.8 to a worksheet posted. The theory can be quite deep (but very interesting).

The main takeaway is its usefulness in

① Seeing where a 1^{st} order ODE is "nice",

② Understanding the intricacies of the theory even at this early stage.

Section 3.1

A general form for a second-order ODE is

$$(*) \qquad y'' = f(t, y, y')$$

for some function f.

<u>Def</u> A 2^{nd} order ODE is called <u>linear</u> if it can be written

$$P(t)\,y'' + Q(t)\,y' + R(t)\,y = G(t)$$

or more familiarly

$$y'' + p(t)\,y' + q(t)\,y = g(t)$$

where $p(t) = \dfrac{Q(t)}{P(t)}$, $q(t) = \dfrac{R(t)}{P(t)}$, $g(t) = \dfrac{G(t)}{P(t)}$

<u>Notes</u> ① Here in (*), this means

$$f(t, y, y') = g(t) - p(t)\,y' - q(t)\,y$$

and f is linear in y and y'.

② If not linear, then non-linear

<u>Def</u> If $g(t) \equiv 0$, then ODE is called <u>homogeneous</u>.

<u>Def</u> An IVP with (*) contains 2 pieces of initial data, usually

$y(t_0) = y_0, \quad y'(t_0) = y'_0$ <u>why?</u>

In general, it is hard or impossible to solve a 2nd order ODE

Even linear is very difficult in general!

One type that is solvable: Constant coefficients

Let (**) have $P(t) \equiv a$, $Q(t) \equiv b$, $R(t) \equiv c$, and suppose $G(t) \equiv 0$ (homogeneous).

Then ODE is $\quad ay'' + by' + cy = 0 \qquad (\star)$

Q: First think, what kinds of functions would possibly be solutions to this kind of ODE?

- Polynomials? Power Functions?
- Trig functions?
- Exponentials?
- Logarithms?

ex Suppose $a=1$, $b=0$, $c=-1$. Then (**) is $y''-y=0$, or $y''=y$.

Solutions? Here $y(t)=e^t$ and $y(t)=e^{-t}$ both solve $y''-y=0$.

How about e^t+e^{-t}? $2e^t-3e^{-t}$?

Here $y(t)=c_1e^t+c_2e^{-t}$ is a solution for any choice of $c_1, c_2 \in \mathbb{R}$.

What if IVP was $y''-y=0$, $y(0)=3$, $y'(0)=4$?

Then

$$\underbrace{\begin{aligned} y(0) &= 3 = c_1e^0+c_2e^{-0} = c_1+c_2 \\ \text{and}\quad y'(0) &= 4 = c_1e^0-c_2e^{-0} = c_1-c_2 \end{aligned}}$$

$$\left.\begin{aligned} 3 &= c_1+c_2 \\ 4 &= c_1-c_2 \end{aligned}\right\} \quad \begin{aligned} c_1 &= \tfrac{7}{2} \\ c_2 &= -\tfrac{1}{2} \end{aligned}$$

And the particular solution to IVP is

$$y(t) = \tfrac{7}{2}e^t - \tfrac{1}{2}e^{-t}.$$

ex 2 $\quad 2y'' + 8y' - 10y = 0$

Here $y(t) = e^{t}$ and $y(t) = e^{-5t}$ both work! Check this....

AND so does $y(t) = c_1 e^{t} + c_2 e^{-5t}$.

Is there a pattern?

For $ay'' + by' + cy = 0$, assume the solution is exponential (is this a good idea?) and looks like $y(t) = e^{rt}$ where r is an unknown parameter.

Then substituting $y(t)$ and its derivatives into the ODE, we get

$$ar^2 e^{rt} + br e^{rt} + c e^{rt} = 0$$

(A) $\quad$ or $\quad ar^2 + br + c = 0$. Any valid values for r must satisfy $r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Recognize this?

For a 2^{nd} order homogeneous ODE w/ constant coefficients, (☆) is called the characteristic equation.

Important fact

When (☆) has 2 solutions r_1, r_2 which are real and distinct ($r_1 \neq r_2$), then the general solution to $ay'' + by' + cy = 0$ is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

(we will show this later).

ex 1. $y'' - y = 0$, with $a = c = 1$, $b = 0$.

characteristic eqn. is $r^2 - 1 = 0$, solved by $r = -1, 1$. Gen. soln is $y(t) = c_1 e^t + c_2 e^{-t}$

ex 2 Characteristic eqn of $2y'' + 8y' - 10y = 0$ is $2r^2 + 8r - 10 = 0 = (2r-2)(r+5)$. Hence gen soln is $y(t) = c_1 e^t + c_2 e^{-5t}$.