Before talking about the cases where roots of the characteristic equation are real or not real, let's return to the more general linear 2nd order homogeneous ODE

\[ y'' + p(t)y' + q(t)y = 0 \]

To study this, form the operator

(An operator is a function whose domain and range are functions)

\[ L[y] = y'' + p(t)y' + q(t)y. \]

This operator is defined for all \( C^2 \) functions \( y(t) \) on an interval like \( a < t < b \), where \( a \) may be a number or \(-\infty\), and \( b \) may be a number or \( \infty \).
Notes

1. Can also write
\[ L = \frac{d^2}{dt^2} + p \frac{d}{dt} + q \]

2. Observe: An operator \( \mathcal{L} \psi_1 \) is linear if
\[ \mathcal{L}[c_1 \psi_1 + c_2 \psi_2] = c_1 \mathcal{L}[\psi_1] + c_2 \mathcal{L}[\psi_2] \]

Consider: \( \mathcal{L}[\psi] = \psi'' + p(t) \psi' + q(t) \psi \)
is linear, as an operator.

\[ \mathcal{L}[c_1 \psi_1 + c_2 \psi_2] = \frac{d^2}{dt^2}[c_1 \psi_1 + c_2 \psi_2] \]

\[ + p(t) \frac{d}{dt}[c_1 \psi_1 + c_2 \psi_2] + q(t)(c_1 \psi_1 + c_2 \psi_2) \]

\[ = c_1 \psi_1'' + c_2 \psi_2'' + p(t)(c_1 \psi_1' + c_2 \psi_2') \]

\[ + q(t)(c_1 \psi_1 + c_2 \psi_2) \]

\[ = c_1 (\psi_1'' + p(t) \psi_1' + q(t) \psi_1) \]

\[ + c_2 (\psi_2'' + p(t) \psi_2' + q(t) \psi_2) \]

\[ = c_1 \mathcal{L}[\psi_1] + c_2 \mathcal{L}[\psi_2] \]
Fact: The homogeneous 2nd order linear ode
\[ y'' + p(t)y' + q(t)y = 0 \]
is solved by any function \( y(t) \), where
\[ L[y(t)] = 0. \]

2 Theorems on linear, 2nd order ODEs

1) Existence & Uniqueness

**Theorem:** The IVP \( y'' + p(t)y' + q(t)y = g(t) \),
\[ y(t_0) = y_0, \quad y'(t_0) = y'_0, \]
where \( p, q, \) and \( g \) are continuous on an open interval \( I \) containing \( t_0 \), has a unique solution \( y(t) \)
defined and twice differentiable on \( I \).

Note: Here, \( I \) can be taken to be the largest interval containing \( t_0 \), where \( p, q, \) and \( g \) are all simultaneously continuous.
Ⅳ Superposition

Then $y_1(t), y_2(t)$ are 2 solutions to $L[y] = 0$, then so is $c_1 y_1 + c_2 y_2$
for all $c_1, c_2 \in \mathbb{R}$.

Caution: Unless $y_1$ and $y_2$ are chosen carefully, there may be other solutions.
We need to know what carefully means here.