Back to the constant coefficients case:

\[(x) \quad ay'' + by' + cy = 0\]

Let \(a = 1, b = 0\). Here \(y'' + y = 0\) has characteristic equation \(r^2 + 1 = 0\) (no real roots).

But, we know \(y_1(t) = \cos t, \ y_2(t) = \sin t\) solve \(y'' + y = 0\) and \(\sin t \ W(\cos t, \sin t) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = 1\).

These subs are independent, and

\[y(t) = C_1 \cos t + C_2 \sin t\]

is a fundamental set of solutions.

Q: How can we get that from the characteristic eqn?

First, \(r^2 + 1 = 0\) has two real solutions: \(r = \pm \sqrt{-1} = \pm i\).

Sticking to the exponential there:

\(y_1(t) = e^{it}, \ y_2(t) = e^{-it}\)

are two solutions. (But they are not real.)

Recall Euler's Formula

\[e^{(a+ib)t} = e^{at}(\cos bt + i \sin bt)\]
Q: Can we construct real solutions from here?

Suppose an ODE has characteristic eqn

$$2r^2 + br + c = 0, \quad r = -\frac{b \pm \sqrt{b^2 - 4ac}}{2a} = \lambda \pm mi, \quad m \neq 0$$

Note: 2 complex roots of a real quadratic polynomial must be conjugates. Why?

Writing 2 complex solns,

$$y_1(t) = e^{(\lambda + mi)t}$$
$$y_2(t) = e^{(\lambda - mi)t}$$

$$= e^{\lambda t}(\cos mt + i\sin mt)$$
$$= e^{\lambda t}(\cos mt - i\sin mt)$$

We see they are not real. But the real ODE must have real solutions!

By superposition, any linear combination of $y_1, y_2$ is also a solution:

Hence

$$\frac{1}{2}(y_1(t) + y_2(t)) = \frac{1}{2}(e^{\lambda t}(\cos mt + i\sin mt) + e^{\lambda t}(\cos mt - i\sin mt))$$

$$= e^{\lambda t}\cos mt \quad \text{is a solution}$$

And

$$\frac{1}{2i}(y_1(t) - y_2(t)) = \sin lt = e^{\lambda t}\sin mt$$

is a solution. \( \text{real!} \)
Let's call these

\[ w(t) = e^{\lambda t} \cos \mu t, \quad v(t) = e^{\lambda t} \sin \mu t \]

Then as \( t \) red shr (the red and imaginary parts of the original apple shrs).

And since \( W(u, v) = \text{calculate this} = \mu e^{2\lambda t} \neq 0 \) everywhere as long as \( \mu \neq 0 \) (making the root apple

Hence, the solution independent.

Hence, in the case of \( ay'' + by' + cy = 0 \) with

determine the equation roots \( \gamma = \lambda + i\mu, \mu \neq 0 \),

The fundamental set of solutions is

\[ y(t) = c_1 e^{\lambda t} \cos \mu t + c_2 e^{\lambda t} \sin \mu t \]

\[ = e^{\lambda t} (c_1 \cos \mu t + c_2 \sin \mu t). \]

Exercise: Check that this is a solution.

Ex. \( y'' + y = 0 \). Roots of \( \gamma^2 + 1 = 0 \) are \( \gamma = \pm i \).

Hence \( \lambda = 0, \mu = 1 \).

\[ y(t) = e^{0t} (c_1 \cos t + c_2 \sin t) = c_1 \cos t + c_2 \sin t. \]
ex. Solve the IVP \( y'' + 4y' + 13y = 0 \)
\[ y(0) = 2, \quad y'(0) = 07. \]

**Solution:** The characteristic equation is \( r^2 + 4r + 13 = 0 \)
\[ a = 4, \quad b = 13 \]
\[ r = \frac{-4 \pm \sqrt{16 - 52}}{2} = -2 \pm 3i \]

Hence the solution is:
\[ y(t) = e^{-2t}(c_1 \cos 3t + c_2 \sin 3t) \]

For a particular solution:
\[ y(0) = e^{-2(0)}(c_1 \cos 3(0) + c_2 \sin 3(0)) = 2 = c_1 \]
\[ y'(0) = -2e^{2t}(2 \cos 3t + 3c_2 \sin 3t) \]
\[ + e^{-2t}(-6 \sin 3t + 3c_2 \cos 3t) \]
\[ = -4 + 3c_2 = 07 \quad c_2 = \frac{07}{3} \]

Particular solution is:
\[ y(t) = e^{-2t}(2 \cos 3t + \frac{07}{3} \sin 3t) \]
Fact: Given $ay'' + by' + cy = 0$ and $r^2 + 5r + c = 0$ with roots $r_1, r_2$.

1. If $r_1 \neq r_2$ real, find set of solns:
   \[ y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \]

2. If $r_1 = r_2$ complex, $r_1 = \lambda + i\alpha \neq \lambda - i\alpha = r_2$:
   \[ y(t) = e^{\lambda t}(c_1 \cos \alpha t + c_2 \sin \alpha t) \]

3. If $r_1 = r_2 = \lambda$ then:
   \[ y(t) = c_1 e^{\lambda t} + c_2 e^{\lambda t} = (c_1 + c_2) e^{\lambda t} = k e^{\lambda t} \]

   Is only 1 solution. We will need another kernel to complete the set of solns.

Q: How