Reduction of order - Using one solution to an \( n \)-th order ODE to create an \( (n-1) \)-th order ODE.

Suppose (\( \ast \) ) \( y'' + p(t)y' + q(t)y = 0 \) has \( y_1(t) \) as a non-zero solution. If you can find an independent second solution, you are done.

Guess: Assume second solution has the form \( y_2(t) = u(t)y_1(t) \)

for some unknown function \( u(t) \).

Why? You will see.

Goal: Try to solve for \( u(t) \).
Now if \( y_2(t) \) solves the ODE, then
\[
\begin{align*}
\dot{y}_2(t) &= \frac{d}{dt} \left[ u(t) y_1(t) \right] = u'(t) y_1(t) + u(t) \dot{y}_1(t) \\
\ddot{y}_2(t) &= u''(t) y_1(t) + u'(t) \dot{y}_1(t) + u(t) \ddot{y}_1(t) + u'(t) \dddot{y}_1(t) \\
&= 2u'(t) y_1'(t)
\end{align*}
\]
Substitute this back into the original ODE:
\[
\left( u'' y_1 + 2u' y_1' + u y_2' \right) + p \left( u y_1' + u y_1' \right) + q y_2' = 0
\]
Recall the terms of \( u \) and derive choices:
\[
y_1 u'' + (2y_1' + py_1) u' + (y_1'' + py_1' + qy_1) u = 0
\]
We are left with:
\[
(A) \quad y_1 u'' + (2y_1' + py_1) u' = 0
\]
This is a 2nd order ODE in \( u'(t) \).
But this is a 1st order ODE in \( u'(t) \)!!
Solve for \( y' \). An interpret \( b \) set \( y \).

Notes:
1. Thinking of \((A)\) as a 1st order ODE in \( y' \), \( t \) is called reducing the order.
2. If the coefficient of \( y' \) weren’t 0, then we cannot do this! Since the lowest order derivative is \( y'' \), we can.
3. For independence,

\[
W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\
         y_1' & y_2' 
   \end{vmatrix} = y_1 y_2' - y_2 y_1' = y_1 y_2 
\]

As long as \( y_1' \neq 0 \), (so \( y_1 \) is not a constant) \( y_2 = y_1 y_1' \) will be independent of \( y_1 \).
ex. \[ y_1(t) = \frac{1}{t} \] solves \[ t^2 y'' + 3ty' + y = 0 \] on the interval \( t > 0 \).

Find a fundamental set of solutions.

**Strategy:** Use reduction of order.

**Solution:** Both \( p(t) = \frac{3}{t} \), \( q(t) = \frac{1}{t^2} \) are \( C^0 \) on \( (0, \infty) \).

Assume \( y_2(t) = v(t)y_1(t) = \frac{v(t)}{t} \).

Then \( v(t) \) solves

\[
\begin{align*}
y_1v'' + (2y'_1 + py_1)v' &= 0, \quad \text{or} \\
\frac{1}{t}v'' + \left( 2\left( -\frac{1}{t^2} \right) + \left( \frac{3}{t} \right) \left( \frac{1}{t} \right) \right)v' &= 0, \quad \text{or} \\
\frac{v''}{t} + \frac{v'}{t^2} &= 0 \Rightarrow tv'' + tv' = \frac{d}{dt} [tv'] = 0 \\
\text{which implies } tv' &= c, \quad \text{a constant, or} \quad v'(t) = \frac{c}{t}, \quad \text{or} \\
v(t) &= c_1 \ln t + c_2 \quad \text{on } t > 0.
\end{align*}
\]

Hence \( y_2(t) = \frac{v(t)}{t} = \frac{c_1 \ln t + c_2}{t} \)
Questions to ask

1. Does \( y_2(t) \) actually solve the original one?

\[
y_2(t) = \frac{c_1 \ln t + c_2}{t}, \quad y_2'(t) = c_1 \left( \frac{1 - \ln t}{t^2} \right) - \frac{c_2}{t^2},
\]

\[
y_2'' = c_1 \left( \frac{2 \ln t - 3}{t^3} \right) + \frac{2c_2}{t^2}
\]

Here \( t^2 y'' + 3ty' + y = 0 = t^2 \left( c_1 \left( \frac{2 \ln t - 3}{t^3} \right) - \frac{2c_2}{t^3} \right) + 3t \left( c_1 \left( \frac{1 - \ln t}{t^2} \right) - \frac{c_2}{t^2} \right) + c_1 \frac{\ln t + c_2}{t} = 0 \)

2. Notice that \( y_1 = \frac{1}{t} \) appears as a summand in

\[
y_2 = c_1 \frac{\ln t + c_2}{t}.
\]

By superposition, it is not really needed. Check independence of \( y_1, y_2 \).

Hence the set of solution is

\[
y(t) = (\text{constant}) y_1 + (\text{constant}) y_2
\]

\[
= (\text{constant}) \frac{1}{t} + (\text{constant}) \left( c_1 \frac{\ln t + c_2}{t} \right)
\]

Combine constants to set

\[
y(t) = \frac{k_1}{t} + k_2 \frac{\ln t}{t}
\]

as the general solution.
Application: Given \( ay'' + by' + cy = 0 \), suppose characteristic eqn has only 1 real soln

\[
\text{Re} \, \gamma = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b}{2a}
\]

Here \( y_1(t) = e^{-\frac{b}{2a}t} \) solves the ODE, but this is the only exponential function that does.

To find another function, reduce the order:

Assume \( y_2 = v(t) e^{-\frac{b}{2a}t} \), where \( v(t) \) solves

\[
v'' + \left(2y_1' + py_1\right)v' = 0, \quad \text{or} \quad e^{-\frac{b}{2a}t}v'' + \left(2\frac{-b}{2a} e^{-\frac{b}{2a}t} + \frac{b}{2} e^{-\frac{b}{2a}t}\right)v' = 0\]

\[
\Rightarrow e^{-\frac{b}{2a}t}v'' = 0 \Rightarrow v'' = 0 \Rightarrow v(t) = k_1t + k_2,
\]

So \( y_2(t) = \left(k_1t + k_2\right)e^{-\frac{b}{2a}t} \)

Exercise: Calculate \( W(y_1, y_2) \) here!
Hence \( y_1(c) \), \( y_2(c) \) form a fundamental set of solutions

\[
y(t) = (\text{constant}) e^{-\frac{1}{2c} t} + (\text{constant}) (kt+k_2) e^{-\frac{1}{2c} t}
\]

\[
y(t) = c_1 e^{-\frac{1}{2c} t} + c_2 t e^{-\frac{1}{2c} t}
\]

**Example:** Solve \( 25y'' - 20y' + 4y = 0 \), \( y(0)=5 \), \( y'(0)=\frac{3}{2} \).

**Solution:** The discriminant \( b^2 - 4ac = 400 - 400 = 0 \)

Hence \( \gamma_1 = \gamma_2 = \gamma = -\frac{1}{2c} = \frac{2}{5} \).

Hence fundamental set of solutions is

\[
y(t) = c_1 e^{\frac{2}{5} t} + c_2 t e^{\frac{2}{5} t}
\]

As for the particular solution:

\[
y(0) = c_1 e^0 + c_2 (0) e^0 = 5 = c_1
\]

So \( y(t) = 5 e^{\frac{2}{5} t} + c_2 t e^{\frac{2}{5} t} \).

And \( y'(t) = 5 \left( \frac{2}{5} \right) e^{\frac{2}{5} t} + c_2 e^{\frac{2}{5} t} + \frac{2c_2}{5} t e^{\frac{2}{5} t} \bigg|_{t=0} = 2 + c_2 = \frac{3}{2} \Rightarrow c_2 = -\frac{1}{2} \)

**Solution:** \( y(t) = 5 e^{\frac{2}{5} t} - \frac{1}{2} e^{\frac{2}{5} t} \).