110.302 Lecture 15: ~~[illegible]~~ I

Let's go back to the original linear, 2nd order ODE

$$(*) \quad L[y] = y'' + p(t)y' + q(t)y = g(t)$$

where $p, q,$ and g are all continuous on some open interval I, and $g(t) \not\equiv 0$ (the non-homogeneous case)

<u>Caution</u>: $(*)$ is linear but superposition does <u>NOT</u> hold here!

<u>Thm</u> Suppose $Y_1(t)$ solves $L[y] = g_1(t)$ and $Y_2(t)$ solves $L[y] = g_2(t)$.

Then $Y_1 + Y_2$ solves $L[y] = g_1(t) + g_2(t)$.

<u>pf.</u> $L[y]$ is linear, so

$$L[Y_1 + Y_2] = L[Y_1] + L[Y_2] = g_1(t) + g_2(t) \quad \blacksquare$$

<u>Corollary</u> Suppose $Y_1(t)$ and $Y_2(t)$ both solve $L[y] = g(t)$. $\Longrightarrow$ $Y_1(t) - Y_2(t)$ solves $L[y] = 0$.

Using this, we can construct solutions to $L[y]=g(t)$.

Let $L[y]=g(t)$ be non-homogeneous, and $Y_1(t)$, $Y_2(t)$ be 2 solutions.

Let $c_1 y_1(t) + c_2 y_2(t)$ be a fundamental set of solutions to the homogeneous $L[y]=0$.

By Corollary, $Y_2(t) - Y_1(t)$ is also a solution to $L[y]=0$, so

① $Y_2 - Y_1 = c_1 y_1 + c_2 y_2$ for some choice of constants $c_1, c_2 \in \mathbb{R}$, and

② $$\underbrace{Y_2}_{\substack{\text{any other}\\ \text{solution to}\\ L[y]=g(t).}} = \underbrace{c_1 y_1 + c_2 y_2}_{\substack{\text{fund set of}\\ \text{solutions to}\\ L[y]=0}} + \underbrace{Y_1}_{\text{a solution to } L[y]=g(t)}$$

We use this to construct a general solution to $L[y]=g(t)$.

Thm The general solution to $L[y] = g(t)$ is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

where y_1, y_2 form a fundamental set of solutions to $L[y] = 0$, and $Y(t)$ is ANY particular solution to $L[y] = g(t)$.

This gives us a method for solving a nonhomogeneous 2^{nd} order linear ODE $L[y] = g(t)$:

I First, solve $L[y] = 0$

II Find any solution to $L[y] = g(t)$

III Put these together to construct the general solution.

The new part here is II, which can be hard.

But there are ways in limited cases.

Here, we highlight 2 ways. (Both involve guessing!)

Undetermined Coefficients.

Suppose $L[y] = g(t)$ has the following form:

① homogeneous part has constant coefficients

② $g(t)$ is a sum of products of

ⓐ exponentials

ⓑ sines and cosines

ⓒ polynomials

Then you can assume a solution $Y(t)$ is of the same type (written out with appropriate unknown coefficients and constants).

Substitute $Y(t)$ into the $L[y] = g(t)$ and try to solve for the coefficients and constants.

---

ex1 $y'' - 2y' - 3y = 3e^{2t}$. Here, a fund. set of solutions to homogeneous part is $c_1 e^{3t} + c_2 e^{-t}$

Assume $Y(t) = Ae^{2t}$. Then $\frac{d^2}{dt^2}(Ae^{2t}) - 2\frac{d}{dt}(Ae^{2t}) - 3(Ae^{2t}) = 3e^{2t}$

$\Rightarrow 4Ae^{2t} - 4Ae^{2t} - 3Ae^{2t} = 3e^{2t}$.

This is solved for $A = -1$. Hence

$\Psi(t) = -e^{2t}$ is a solution to $L[y] = 3e^{2t}$.

Thus the general solution to $y'' - 2y' + 3y = 3e^{2t}$

is $y(t) = c_1 e^{3t} + c_2 e^{-t} - e^{2t}$

---

ex2 Solve $y'' - 2y' - 3y = 3\sin 3t$

Here homogeneous part is same as ex1. Assume

$\Psi(t) = A\sin 3t + B\cos 3t$. (Why both?

Because of derivatives Ψ' and Ψ''!)

Then

$$\frac{d^2}{dt^2}(\Psi(t)) - 2\frac{d}{dt}(\Psi(t)) - 3\Psi(t) = 3\sin 3t$$

$$-9A\sin 3t - 9B\cos 3t - 2(3A\cos 3t - 3B\sin 3t) - 3(A\sin 3t + B\cos 3t) = 3\sin 3t$$

Here there are 2 equations to solve:

$$\begin{matrix}\text{sine eqn:} & -9A + 6B - 3A = 3 \\ \text{cosine eqn:} & -9B - 6A - 3B = 0\end{matrix} \Bigg\} \quad \begin{cases} -12A + 6B = 3 \\ -6A - 12B = 0 \end{cases}$$

Solved by $A = -\frac{1}{5}$, $B = \frac{1}{10}$.

Hence fund set of solns is

$$y(t) = c_1 e^{3t} + c_2 e^{-t} - \frac{1}{5}\sin 3t + \frac{1}{10}\cos 3t$$

<u>There are many warnings here</u>.

The chart on page 182 gives the general rules for constructing the assumption $Y(t)$ for given $g(t)$.

<u>Notes</u> ① When $g(t)$ actually looks like one of the pieces of the fundamental set of solutions to $L[y]=0$, one must change $Y(t)$ accordingly.

② If RHS $g(t)$ includes a polynomial, one must include unknown constants for every intermediate degree monomial.

③ Be careful of the s. When $g(t)$ has a piece that looks like one of the fund set of solns to $L[y]=0$, one must mult by t^s where s is the smallest positive power that removes the problem.