Lecture 15: 

Let's go back to the original linear, 2nd order ODE:

\[ L[y] = y'' + p(t)y' + q(t)y = g(t) \]

where \( p, q, \) and \( g \) are all continuous on some open interval \( I \), and \( g(t) \neq 0 \) (the non-homogeneous case).

Cauchy: \( (x) \) is linear, but superposition does not hold here.

Then suppose \( \Sigma_1(t) \) solves \( L[y] = g_1(t) \) and \( \Sigma_2(t) \) solves \( L[y] = g_2(t) \).

Then \( \Sigma_1 + \Sigma_2 \) solves \( L[y] = g_1(t) + g_2(t) \).

Proof: \( L[y] \) is linear, so

\[ L[\Sigma_1 + \Sigma_2] = L[\Sigma_1] + L[\Sigma_2] = g_1(t) + g_2(t) \]

Corollary: Suppose \( \Sigma_1(t) \) and \( \Sigma_2(t) \) both solve \( L[y] = g(t) \). \( \Rightarrow \) \( \Sigma_1(t) - \Sigma_2(t) \) solves \( L[y] = 0 \).
Using this, we can construct solutions to \( L[y] = g(x) \).

Let \( L[y] = g(x) \) be non-homogeneous, and \( \Sigma_1(x), \Sigma_2(x) \) be 2 solutions.

Let \( c_1 \Sigma_1(x) + c_2 \Sigma_2(x) \) be a fundamental set of solutions to the homogeneous \( L[y] = 0 \).

By Corollary, \( \Sigma_2(x) - \Sigma_1(x) \) is also a solution to \( L[y] = 0 \), so

1. \( \Sigma_2 - \Sigma_1 = c_1 \Sigma_1 + c_2 \Sigma_2 \) for some choice of constants \( c_1, c_2 \in \mathbb{IR} \), and

2. \( \Sigma_2 = \underbrace{c_1 \Sigma_1 + c_2 \Sigma_2 + \Sigma_1}_{\text{any other solution to } L[y] = g(x)} \) a solution to \( L[y] = g(x) \).

We use this to construct a general solution to \( L[y] = g(x) \).
Then the general solution to $Ly = g(t)$ is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \Phi(t)$$

where $y_1, y_2$ form a fundamental set of solutions to $Ly = 0$, and $\Phi(t)$ is any particular solution to $Ly = g(t)$.

This gives us a method for solving a nonhomogeneous 2nd order linear ODE $Ly = g(t)$:

1. First, solve $Ly = 0$
2. Find any solution to $Ly = g(t)$
3. Put these together to construct the general solution.

The new part here is 2, which can be hard.

But there are ways in limited cases. Here, we highlight 2 ways. (Both involve guessing...)
Undetermined Coefficients.

Suppose \( L[y] = g(t) \) has the following form:

1. **Homogeneous part has constant coefficients**
2. \( g(t) \) is a sum of products of:
   - exponentials
   - sines and cosines
   - polynomials

Then you can assume a solution \( \Sigma(t) \) is of the same type (written out with appropriate unknown coefficients and constants).

Substitute \( \Sigma(t) \) into \( L[y] = g(t) \) and try to solve for the coefficients and constants.

\[
\exp 1 y'' - 2y' - 3y = 3e^{2t}.
\]

Here, a fund set of solutions to homogeneous part is \( e_1 e^{2t} + e_2 e^{-t} \)

Assume \( \Sigma(t) = A e^{2t} \). Then \( \frac{d^2}{dt^2} (A e^{2t}) + 2 \frac{d}{dt} (A e^{2t}) - 3(A e^{2t}) \)

\[
= 3e^{2t}.
\]

\[
\Rightarrow 4A e^{2t} - 4A e^{2t} - 3A e^{2t} = 3e^{2t}.
\]
This is solved for $A = -1$. Hence
\[ y(t) = -e^{2t} \] is a solution to $y'' - 2y' + 3y = 3e^{2t}.$

Thus the general solution to $y'' - 2y' + 3y = 3e^{2t}$ is
\[ y(t) = c_1 e^{3t} + c_2 e^{-t} - e^{2t} \]

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**Exercise:** Solve $y'' - 2y' - 3y = 3 \sin 3t$

Here homogeneous part is same as $e^{kt}$. Assume
\[ y(t) = A \sin 3t + B \cos 3t. \] (Why so?)

Because of derivatives $y'$ and $y''$!

Then
\[
\frac{d^2}{dt^2} (y(t)) - 2 \frac{dy}{dt} (y(t)) - 3y(t) = 3 \sin 3t
\]
\[ -9A \sin 3t - 9B \cos 3t - 2(3A \cos 3t - 3B \sin 3t) - 3(A \sin 3t + B \cos 3t) = 3 \sin 3t \]

Here there are 2 equations to solve:

\[
\begin{align*}
\sin e & : & -9A + 6B - 3A = 3 & \iff -12A + 6B = 3 \\
\cos e & : & -9B - 6A - 3B = 0 & \iff -6A - 12B = 0 \\
\end{align*}
\]

Solved by $A = -\frac{1}{3}$, $B = \frac{1}{10}$.

Hence the solution is
\[ y(t) = c_1 e^{3t} + c_2 e^{-t} - \frac{1}{3} \sin 3t + \frac{1}{10} \cos 3t. \]
Here are many warnings here.

The chart on page 132 gives the general rules for constructing the assumption \( \Gamma(t) \) for given \( g(t) \).

**Notes**

1. When \( g(t) \) actually looks like one of the pieces of the fundamental set of solutions to \( L[y] = 0 \), one must choose \( \Gamma(t) \) accordingly.

2. If \( L[y] \) includes a polynomial, one must include unknown constants for every intermediate degree monomial.

3. Be careful of the \( s \). When \( g(t) \) has a piece that looks like one of the third set of solutions to \( L[y] = 0 \), one must multiply by \( t^s \) where \( s \) is the smallest positive power that removes the problem.