110.302 Lecture 16:

<u>ex 3</u> $y'' - 2y' - 3y = 4e^{-t}$

Again, fund. set of solutions to $y'' - 2y' - 3y = 0$

is $c_1 e^{3t} + c_2 e^{-t}$.

Caution: Here -1 is a root of char. eqn. of

homogeneous part so we cannot assume

$$Y(t) = Ae^{-t}$$

It is already part of $c_1 e^{3t} + c_2 e^{-t}$.

We fix this by setting $s = 1$, and $Y(t) = Ate^{-t}$

<u>ex 4</u> $y'' - 4y' + 4 = 12e^{2t}$. Here $r = 2$ is the only solution to $r^2 - 4r + 4 = 0$

Fund. set of solns to homogeneous part is

$c_1 e^{2t} + c_2 t e^{2t}$.

Here, $g(t) = 12e^{2t}$, so assume $Y(t) = At^2 e^{2t}$

($s = 2$ since $r = 2$ is a double root to $r^2 - 4r + 4$.)

ex 5 $y'' - 4y' + 4y = 3t^3 e^{-2t}$

Here $Y(t) = t^2(At^3 + Bt^2 + Ct + D)\, e^{2t}$

This method is useful but limited in scope

① LHS must have constant coefficients

② RHS must be nice

Here is a more general idea: Variation of parameters

① Is a form of reduction of order

② Works for any 2^{nd} order linear nonhomogeneous ODE

③ Relies on two assumptions.

Given $y'' + p(t)y' + q(t)y = g(t)$, suppose $c_1 y_1(t) + c_2 y_2(t)$ is a fundamental set of solutions to $L[y] = 0$.

Assumption 1 Assume $\boxed{\Psi(t) = u_1(t) y_1(t) + u_2(t) y_2(t)}$ solves $L[y] = g(t)$, for u_1, u_2 unknown fncs. (Compare to reduction of order technique).

Then $\Psi'(t) = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'$

Note: This is messy, but a good assumption. We can make this easier to handle.

Assumption 2 Assume $\boxed{u_1' y_1 + u_2' y_2 = 0}$

Then $\Psi'(t) = u_1 y_1' + u_2 y_2'$, and

$$\Psi''(t) = u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2''$$

Substitute these into $L[y] = g(t)$ and get

$$(u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2'') + p(u_1 y_1' + u_2 y_2') + q(u_1 y_1 + u_2 y_2) = g(t)$$

Rearrange to get

$$u_1 \underbrace{(y_1'' + p y_1' + q y_1)}_{0} + u_2 \underbrace{(y_2'' + p y_2' + q y_2)}_{0} + u_1' y_1' + u_2' y_2' = g(t)$$

$$\boxed{u_1' y_1' + u_2' y_2' = g(t)}$$

Here, Assumption 2 is a good one since

ⓐ First assumption allows a lot of freedom since 2 unknowns functions are present.

ⓑ Second assumption allows for no second derivatives of u_1, u_2 in ODE.

Both assumptions, within ODE yield the system

$$u_1' y_1 + u_2' y_2 = 0$$
$$u_1' y_1' + u_2' y_2' = g(t)$$

Solve this for u_1' and u_2', integrate each to find $u_1(t)$ and $u_2(t)$. Are there solutions?

Solving, we get

$$u_1' = \frac{-y_2 g}{y_1 y_2' - y_2 y_1'} = \frac{-y_2 g}{W(y_1, y_2)}$$

$$u_2' = \frac{y_1 g}{y_1 y_2' - y_2 y_1'} = \frac{y_1 g}{W(y_1, y_2)}$$

Hence

$$u_1 = \int \frac{-y_2 g}{W(y_1, y_2)}\, dt, \qquad u_2 = \int \frac{y_1 g}{W(y_1, y_2)}\, dt$$

With these, $\Upsilon(t) = u_1 y_1 + u_2 y_2$ is one particular solution, and

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \Upsilon(t)$$

is the general solution to $y'' + p y' + q y = g$

What does this look like in practice?

ex 3.6.14 Knowing $y_1(t) = t$, and $y_2(t) = te^t$ both solve $t^2y'' - t(t+2)y' + (t+2)y = 0$ on $t > 0$, find the general solution to

$$t^2y'' - t(t+2)y' + (t+2)y = 2t^3.$$

Strategy: We use the Variation of Parameters method with

$$Y(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t) = u_1 t + u_2 te^t$$

Note: Here, $g(t) = 2t$, not $2t^3$

Solution: Given this assumption for $Y(t)$, we obtain the system $u_1'y_1 + u_2'y_2 = 0$, $u_1'y_1' + u_2'y_2' = g(t)$,

$$\left.\begin{aligned} u_1't + u_2'te^t &= 0 \\ u_1' + u_2'(e^t + te^t) &= 2t \end{aligned}\right\} \begin{array}{c}(-t) * \text{eqn } 2 \\ \text{and add}\end{array} \left\{\begin{aligned} u_1't + u_2'te^t &= 0 \\ -u_1't - u_2't(e^t + te^t) &= -2 \end{aligned}\right.$$

Add eqn 1 to eqn 2 to get $-u_2't^2e^t = -2t^2$

or $u_2' = 2e^{-t}$, so $\boxed{u_2(t) = -2e^{-t}}$

Then, using eqn 1, $u_1't + (2e^{-t})te^t = 0$, or $u_1' = -2$

so $\boxed{u_1(t) = -2t}$

Then $Y(t) = -2t(t) + (-2e^{-t})te^t = -2t^2 - 2t$

So general solution is $y(t) = c_1t + c_2te^t - 2t^2 - 2t$

or $\boxed{y(t) = k_1 t + c_2 t e^t - 2t^2}$

Note: We could appeal directly to the general form for u_1, u_2: $W(t, te^t) = \begin{vmatrix} t & te^t \\ 1 & e^t + te^t \end{vmatrix} = t^2 e^t$ on $t > 0$

$$u_1(t) = \int \frac{-y_2 g}{W(y_1, y_2)}\,dt = \int \frac{-(te^t)\,2t}{t^2 e^t}\,dt = -\int 2\,dt = -2t$$

$$u_2(t) = \int \frac{y_1 g}{W(y_1, y_2)}\,dt = \int \frac{t(2t)}{t^2 e^t}\,dt = 2\int e^{-t}\,dt = -2e^{-t}$$

<u>Question</u>: So why are there no constants of integration here?