The n-th order version of a linear ODE is

\[ a_n(t) y^{(n)} + a_{n-1}(t) y^{(n-1)} + \ldots + a_1(t) y' + a_0(t) y = g(t) \]

which can also be written like an operator

\[ L[y] = y^{(n)} + \rho_n(t) y^{(n-1)} + \rho_{n-1}(t) y^{(n-2)} + \ldots + \rho_1(t) y' + \rho_0(t) y = g(t) \]

where we divide each of the coefficients in the top description by \( a_n(t) \) (e.g. \( \rho_n(t) = \frac{a_{n-1}(t)}{a_n(t)} \))

The theory generalizes in the obvious ways:

1. If the ODE is an IVP, then we will need n pieces of information to completely determine a solution (think n integration to get solution creating an n-parameter family of functions:

\[ y(t_0) = y_0, \quad y'(t_0) = y'_0, \quad \ldots \]

\[ \ldots, \quad y^{(n-2)}(t_0) = y^{(n-2)}_0, \quad y^{(n-1)}(t_0) = y^{(n-1)}_0. \]
II \[\text{Thm (Existence and Uniqueness)}\]

In \((*)\), if \(p_1, \ldots, p_n\) are all continuous on some common interval \(I\), then there exists a unique solution to \((*)\) passing through any set of initial values \(y(0)\) to \(y(n)\).

III \[\text{Superposition Holds: if } y_1(t) \text{ and } y_2(t) \text{ both solve } L[y] = 0 \text{ (homogeneous)} \]

where \(L[y]\) corresponds to the \(n\)th order ODE \((*)\), then \(c_1 y_1(t) + c_2 y_2(t)\) is also a solution.

IV \[\text{GIVEN n solutions } y_1, \ldots, y_n \text{ to an } \]

\(n\)th order homogeneous \(L[y] = 0\), if

\[
W(y_1, \ldots, y_n)(t) = \begin{vmatrix} y_1(t) & y_2(t) & \cdots & y_n(t) \\ y_1'(t) & y_2'(t) & \cdots & y_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \cdots & y_n^{(n-1)}(t) \end{vmatrix}
\]

is nonzero at \(t \in I\), then every solution to the ODE is a linear combination of \(y_1, \ldots, y_n\)
In this case, a fundamental set of solutions,\[ u \text{ s.t. } y(t) = c_1y_1(t) + \ldots + c_ny_n(t) \]

\( \Box \) In fact, it can be shown that for any choice of \( y_1, \ldots, y_n \) solves to \( (\Phi) \),
\[ \Omega(y_1, \ldots, y_n) = C e^{-\int p_i(t) dt} \]
and \( u \) either always 0 or I when \( p_i(t) \) is curt, or never 0 or I.

\( \Box \) If \( a(t), q(t) \neq 0 \), then a general solution is the same as that of a 2nd order non-homogeneous ODE:
\[ y(t) = c_1y_1(t) + \ldots + c_ny_n(t) + \underbrace{\text{fund set of}} \]
\[ \text{solution to } \quad \begin{cases} \Omega & \text{ periccular} \\ \Omega y = 0 & \end{cases} \]
\[ y(t) \]
Q: How to solve the nth-order linear ODE?

\[ a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \ldots + a_1(t)y' + a_0(t)y = q(t) \]

or

\[ y^{(n)} + p_n(t)y^{(n-1)} + \ldots + p_1(t)y' + p_0(t)y = q(t) \]

A: Same as before is the short answer:

The homogeneous part \((L[y] = 0)\), if the coefficients are constants, can be solved by exponentials: Assume \( y(t) = e^{rt} \) to construct the characteristic eqn:

\[ a_n t^n + a_{n-1} t^{n-1} + \ldots + a_1 t + a_0 = 0 \]

The roots of \((*)\) correspond to solutions \( y(t) = e^{rt} \) which are solutions to \( L[y] = 0 \).

The rest of the theory holds also:

1. If \((*)\) can be found (all of them, counting multiplicity and complex conjugates), one can construct an \( n \)-parameter family of solutions (the fundamental set of solutions).
Ex. Suppose \((x)\) has all real distinct roots
\[ r_1 \neq r_2 \neq \ldots \neq r_n \; \text{then}\]
\[ y(x) = c_1 e^{r_1 x} + \ldots + c_n e^{r_n x}\]
is the general solution.

Ex. For repeated roots, the pattern is similar to the 2nd order version.

2) Suppose characteristic eqn of a 5th order OD\(E\) were
\[ (\gamma - 2)(\gamma + 1)(\gamma - 5) = 0 \]
Hence \(r_1 = 2, \; r_2 = r_3 = r_4 = -1, \; r_5 = 5\), and
\[ y(x) = c_1 e^{2x} + c_2 e^{-x} + c_3 x e^{-x} + c_4 x^2 e^{-x} + c_5 e^{5x} \]

3) Suppose \((\gamma - 6)(\gamma^2 - 45 + 12)^2 = 0\)
\[ \Rightarrow r_1 = \sqrt{6}, \; r_2 = -\sqrt{6}, \; r_3 = r_4 = 2 + 3i \]
\[ r_5 = r_6 = 2 - 3i \]
and
\[ y(x) = c_1 e^{\sqrt{6}x} + c_2 e^{-\sqrt{6}x} + e^{2x}(c_3 \cos 3x + c_4 \sin 3x) + 3e^{2x}(c_5 \cos 3x + c_6 \sin 3x) \]

Solution methods for non-homogeneous linear nth order ODEs are the same:

1) Undetermined Coefficients - exactly the same as the 2nd order version
2) Variation of Parameters