110.302 Lecture 19:

The nth-order version of a linear ODE is

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y = G(t)$$

which ~~may~~ can also be written like an operator

$$L[y] = y^{(n)} + p_1(t)\,y^{(n-1)} + p_2(t)\,y^{(n-2)} + \cdots + p_{n-1}(t)y' + p_n(t)y = g(t) \qquad (*)$$

where we divide each of the coefficients in the top description by $a_n(t)$ (ex. $p_1(t) = \frac{a_{n-1}(t)}{a_n(t)}$)

The theory generalizes in the obvious ways:

(I) If the ODE is an IVP, then we will need n pieces of information to completely determine a solution (think n integrations to get solution creating an n-parameter family of functions:

$$y(t_0) = y_0, \quad y'(t_0) = y_0', \quad \ldots$$
$$\ldots, y^{(n-2)}(t_0) = y_0^{(n-2)}, \quad y^{(n-1)}(t_0) = y_0^{(n-1)}.$$

Thm (Existence and Uniqueness)

II. In (*), if $p_1, \cdots, p_n, g$ are all continuous on some common interval I, then there exists a unique solution to (*) passing through any set of initial values @ $t_0 \in I$.

III. Superposition Holds: if $y_1(t)$ and $y_2(t)$ both solve $L[y] = 0$ (homogeneous) where $L[y]$ corresponds to the nth order ODE (*), then $c_1 y_1(t) + c_2 y_2(t)$ is also a solution.

IV. Given n solutions $y_1, \cdots, y_n$ to an nth order homogeneous $L[y] = 0$, if

$$W(y_1, \cdots, y_n)(t) = \begin{vmatrix} y_1(t) & y_2(t) & \cdots & y_n(t) \\ y_1'(t) & y_2'(t) & \cdots & y_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \cdots & y_n^{(n-1)}(t) \end{vmatrix}$$

Wronskian

is nonzero at $t \in I$, then every solution to the ODE is a linear combination of $y_1, \cdots, y_n$.

In this case, a fundamental set of solutions. then is

$$y(t) = c_1 y_1(t) + \cdots + c_n y_n(t)$$

V) In fact, it can be shown that for any choice of $y_1, \cdots, y_n$ solns to $(*)$,

$$W(y_1, \cdots, y_n) = c\, e^{-\int p_1(t)\,dt}$$

and is either always 0 on I where $p_1(t)$ is cont, or never 0 on I.

VI) if in $(*)$, $g(t) \not\equiv 0$, then a general solution is the same as that of a 2nd order nonhomogeneous ODE:

$$y(t) = \underbrace{c_1 y_1(t) + \cdots + c_n y_n(t)}_{\substack{\text{fund set of} \\ \text{solutions to} \\ L[y]=0}} + \underbrace{Y(t)}_{\substack{\text{any particular} \\ \text{solution to} \\ L[y]=g(t)}}$$

110.302 Lecture 18: [illegible]

Q: How to solve the nth-order linear ODE?

$$\underbrace{a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y}_{L[y]} = G(t)$$

or

$$\overbrace{y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y}^{L[y]} = g(t)$$

A: Same as before is the short answer:

The homogeneous part ($L[y]=0$), if the coefficients are constants, can be solved by exponentials: Assume $L[e^{rt}]=0$ to construct the characteristic eqn:

$$(*) \quad a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0$$

The roots of $(*)$ correspond to solutions $y(t) = e^{rt}$ which are solutions to ~~the~~ $L[y]=0$.

The rest of the theory holds also:

① If roots of $(*)$ can be found (all of them, counting multiplicity and complex conjugates, one can construct an n-parameter family of solutions (the fund. set of solns: [illegible]

~~With the exception for the other cases~~

ex. Suppose (*) has all real distinct roots
$r_1 \neq r_2 \neq \cdots \neq r_n$: Rea
$$y(t) = c_1 e^{r_1 t} + \cdots + c_n e^{r_n t}$$
is the general solution.

ex. For repeated roots, the pattern is similar to the 2nd order version

ⓐ Suppose characteristic eqn of a 5^{th} order ODE were $(r-2)(r+1)^3(r-5) = 0$

then $r_1 = 2$, $r_2 = r_3 = r_4 = -1$, $r_5 = 5$, and
$$y(t) = c_1 e^{2t} + c_2 e^{-t} + c_3 t e^{-t} + c_4 t^2 e^{-t} + c_5 e^{5t}$$

ⓑ Suppose $(r^2-6)(r^2-4r+13)^2 = 0$
$$\Rightarrow r_1 = \sqrt{6}, \ r_2 = -\sqrt{6}, \ r_3 = r_5 = 2+3i$$
$$r_4 = r_6 = 2-3i$$

and $y(t) = c_1 e^{\sqrt{6}t} + c_2 e^{-\sqrt{6}t} + e^{2t}(c_3 \cos 3t + c_4 \sin 3t) + t e^{2t}(c_5 \cos 3t + c_6 \sin 3t)$

Solution methods for non-homogeneous linear nth order ODEs are the same:

① Undetermined Coefficients - exactly the same as the 2nd order version

② Variation of Parameters