Lecture 18

Continuing w/ nth order linear ODEs in studying both structure and patterns and methods of solution and study.

Recall that the characteristic eqn for

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

is

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0,$$

and will always have n solutions if counted w/ multiplicity and allowing complex solutions.

And all intuitive patterns still hold:

ex 1 Suppose char. eqn is $(r^2-6)(r^2-4r+13)^2 = 0$

$$\Rightarrow r_1 = \sqrt{6},\ r_2 = -\sqrt{6},\ r_3 = r_5 = 2+3i$$
$$r_4 = r_6 = 2-3i$$

Then general soln is $y(t) = c_1 e^{\sqrt{6}t} + c_2 e^{-\sqrt{6}t}$ what the t?

$$+ e^{2t}(c_3 \cos 3t + c_4 \sin 3t) + t e^{2t}(c_5 \cos 3t + c_6 \sin 3t)$$

Solution methods for linear nth order ODEs are the same, suitably generalized:

(I) Reduction of Order works well: if $y_1(t)$ solves a linear nth order ODE, then $y_2(t) = v(t) y_1(t)$, as another solution, leads to an $(n-1)$th order ODE in v'.

exercise: Derive the 2nd order ODE in v' using Reduction of Order on

$$y''' + p y'' + q y' + r y = 0$$

where $y_1(x)$ is already a solution.

Hint: This will be a homework problem, but the solution is

$$(*) \quad y_1 v''' + (3y_1' + p y_1) v'' + (3y_1'' + 2p y_1' + q y_1) v' = 0.$$

(II) For finding nonhomogeneous solutions, the Method of Undetermined Coefficients is exactly the same

ex2 The ODE

$$y^{(6)} - 8y^{(5)} + 36y^{(4)} - 56y^{(3)} - 83y'' + 624y' - 1014y = 3e^{2t}\cos 3t$$

has characteristic eqn

$$(r^2-6)(r^2-4r+13)^2 = 0 \quad (\text{from ex1})$$

What is a good guess for $Y(t)$?

(III) For Variation of Parameters,

assume $Y(t) = u_1 y_1 + \cdots + u_n y_n$, for $y_1, \cdots, y_n$ homogeneous solns.

Rem. 16 we play the same game, taking derivatives, making simplifying assumptions along the way, and plugging in the derivatives of Y into the ODE, we arrive at the system of n equations.

$$\begin{aligned}
u_1' y_1 + & \cdots + u_n' y_n = 0 \\
u_1' y_1' + & \cdots + u_n' y_n' = 0 \\
u_1' y_1'' + & \cdots + u_n' y_n'' = 0 \\
\vdots \quad & \ddots \quad \vdots \\
u_1' y^{(n-1)} + & \cdots + u_n' y^{(n-1)} = g(t)
\end{aligned}$$

Solve this for the n-unknowns $u_1', \dots, u_n'$ and then integrate to find $u_1, \dots, u_n$ to form $Y(t) = u_1 y_1 + \cdots + u_n y_n$.

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ex 3: Knowing that $y_1(x) = x$ and $y_2(x) = x^2$ both solve

$$x^3 y''' + x^2 y'' - 2xy' + 2y = 0, \quad \text{on } x > 0$$

solve the ODE

$$x^3 y''' + x^2 y'' - 2xy' + 2y = 2x^4 \quad \text{on } x > 0$$

Strategy: Use Reduction of Order to find the third homogeneous soln. Then use Variation of Parameters to solve the nonhomogeneous ODE.

Solution: In standard form, the ODE is

$$y''' + \frac{1}{x}y'' - \frac{2}{x^2}y' + \frac{2}{x^3}y = 2x,$$

so $p = \frac{1}{x}$, $q = \frac{-2}{x^2}$, $r = \frac{2}{x^3}$, $g = 2x$.

With $y_1 = x$, so $y_1' = 1$, $y_1'' = 0$, we use (ex 1) hint to arrive at our RofO 2nd order ODE in v':

(*) $$y_1 v''' + (3y_1' + py_1)v'' + (3y_1'' + 2py_1' + qy_1)v' = 0$$

$$x v''' + \left(3(1) + \left(\tfrac{1}{x}\right)x\right)v'' + \left(3(0) + \underbrace{2\left(\tfrac{1}{x}\right)(1) + \left(\tfrac{-2}{x^2}\right)x}_{0}\right)v' = 0$$

(A) $$x v''' + 4v'' + 0 = 0$$

Note: This is actually a 1st order ODE in v''.

One immediate soln is $v'' = 0$, which leads to $v(t) = x$ but then $Y(t) = vy_1 = x^2$ which is y_2.

But (A) is also linear 1st order in v'', on $x > 0$,

$$v''' + \frac{4}{x}v'' = 0$$

with integrating factor $e^{\int \frac{4}{x}dx} = \cdots = x^4$.

So $$x^4\left[v''' + \frac{4}{x}v'' = 0\right]$$

$$\underbrace{x^4 v''' + 4x^3 v''}= 0$$

$$\frac{d}{dx}\left[x^4 v''\right] = 0$$

Integrate to get $v = \frac{K}{x^2}$ for K a constant.

(there are other terms but they are parts of the other homogeneous solns).

Thus $y_3(x) = v y_1 = \frac{K}{x^2} \cdot x = \frac{K}{x}$. The K does not matter also, and thus we get

$$c_1 x + c_2 x^2 + c_3\left(\frac{1}{x}\right)$$

as a fundamental set of solns to the ODE

$$y''' + \frac{2}{x}y'' - \frac{2}{x^2}y' + \frac{2}{x^3} = 0$$

To find a nonhomogeneous soln, we can proceed directly to the system

$$u_1' y_1 + u_2' y_2 + u_3' y_3 = 0$$
$$u_1' y_1' + u_2' y_2' + u_3' y_3' = 0$$
$$u_1' y_1'' + u_2' y_2'' + u_3' y_3'' = 2x, \quad \text{or}$$

$$u_1' x + u_2' x^2 + u_3'\left(\tfrac{1}{x}\right) = 0 \Rightarrow \boxed{u_1' x^2 + u_2' x^3 + u_3' = 0} \quad (1)$$
$$u_1' + u_2'(2x) + u_3'\left(-\tfrac{1}{x^2}\right) = 0 \Rightarrow \boxed{u_1' x^2 + u_2'(2x^3) - u_3' = 0} \quad (2)$$
$$u_2'(2) + u_3'\left(\tfrac{2}{x^3}\right) = 2x \Rightarrow \boxed{u_2'(x^3) + u_3' = x^4} \quad (3)$$

- $-(1)+(2). \Rightarrow u_2'(x^3) - u_3'(2) = 0.$
- Pair this with (3). $\Rightarrow$
$$u_2'(x^3) - u_3'(2) = 0 \quad ((1))$$
$$u_2'(x^3) + u_3' = x^4 \quad ((2))$$
- $((1)) - ((2)). \Rightarrow -u_3'(3) = -x^4 \Rightarrow u_3' = \frac{x^4}{3} \Rightarrow u_3 = \frac{x^5}{15}$
- Sub back into (1) & (2) and pair:
$$(((1)))\quad u_1' x^2 + u_2' x^3 = -x^4$$
$$(((2)))\quad u_1' x^2 + u_2'(2x^3) = x^4$$
- $(((1))) - (((2)))$: $-u_2' x^3 = -2x^4 \Rightarrow u_2' = 2x, \quad u_2 = x^2$
- Sub back into (1): $u_1' x^2 + (2x)x^3 + x^4 = 0 \Rightarrow u_1' = -3x^2, \Rightarrow u_1 = -x^3$

So $Y(x) = u_1 y_1 + u_2 y_2 + u_3 y_3$

$$= \cancel{0}\,x^3(x) + x^2(x^2) + \frac{x^5}{15}\left(\frac{1}{x}\right) = \frac{1}{15}x^4$$

And the general soln is

$$y(x) = c_1 x + c_2 x^2 + c_3\left(\frac{1}{x}\right) + \frac{1}{15}x^4.$$