

110.302 Lecture 19: ~~XXXXXXXXXXXX~~

I

Consider the "system" of 2 1st order ODEs in 2 variables:

$$\left. \begin{aligned} \frac{dx}{dt} &= a_1 x - b_1 x y \\ \frac{dy}{dt} &= -a_2 y + b_2 x y \end{aligned} \right\} \begin{array}{l} a_1, a_2, b_1, b_2 > 0 \\ \text{constants.} \end{array}$$

Here, both $x(t)$, $y(t)$ are fncs. of time, and their evolution (derivatives) are intertwined (coupled).

Many applications appear this way. These are called the Lotka-Volterra equations: model the population size of 2 species in a closed environment (predator-prey)

A solution is a set of expressions for $x(t)$ and $y(t)$ that satisfy both equations.

Q: Say $x(t)$ and $y(t)$ represent rabbits and foxes (not necessarily respectively). Can you tell from the ODE system which is which? How?

Why study systems of coupled equations?

2 reasons:

① Many apps appear this way. There are many measurable quantities all depending on a single independent variable (not like vector calculus). In general, this looks like

$$(*) \quad \left. \begin{aligned} \dot{x}_1 &= F_1(t, x_1, \dots, x_n) \\ \dot{x}_2 &= F_2(t, x_1, \dots, x_n) \\ &\vdots \\ \dot{x}_n &= F_n(t, x_1, \dots, x_n) \end{aligned} \right\} \begin{array}{l} \text{1st order} \\ \text{system of} \\ \text{ODEs.} \end{array}$$

where x_1, \dots, x_n are the set of dependent variables and time t is the independent var.

② Any higher order ODE can be transformed (rewritten) as a system of 1st order ODEs:

$$\text{Let } y^{(n)} = F(t, y, y', \dots, y^{(n-1)}).$$

Given the new vars:

$$x_1 = y, \quad x_2 = y', \quad x_3 = y'', \quad \dots, \quad x_n = y^{(n-1)},$$

we get

$$\left. \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= F(t, x_1, \dots, x_n) \end{aligned} \right\} \begin{array}{l} \dot{x}_1 = \dot{y} = y' = x_2 \\ \dot{x}_2 = (y')' = y'' = x_3 \\ \vdots \\ \dot{x}_n = (y^{(n-1)})' = y^{(n)} = F. \end{array}$$

A solution to (*) is a set of functions

$$x_1(t), x_2(t), \dots, x_n(t)$$

If initial values are specified, we would need 1 data pt for each $x_i(t)$, $i=1, \dots, n$

$$x_1(t_0) = x_1^0, \dots, x_n(t_0) = x_n^0.$$

This is identical to

$$y(t_0) = y_0, \dots, y^{(n-1)}(t_0) = y_0^{(n-1)}$$

n-bits of initial data.

Existence and uniqueness for 1st order systems
(similar to that for a single eqn)

Thm In (*), let F_1, \dots, F_n and all of

$$\frac{\partial F_1}{\partial x_1}, \dots, \frac{\partial F_1}{\partial x_n}, \frac{\partial F_2}{\partial x_1}, \dots, \frac{\partial F_2}{\partial x_n}, \dots, \frac{\partial F_n}{\partial x_1}, \dots, \frac{\partial F_n}{\partial x_n}$$

(all partials wrt dep vars x_i , but not t)

be continuous in a region R of the $(n+1)$ -dim t, x_1, \dots, x_n -space defined by

$$\alpha < t < \beta, \alpha_1 < x_1 < \beta_1, \dots, \alpha_n < x_n < \beta_n,$$

and let $p = (t_0, x_1^0, \dots, x_n^0) \in \mathbb{R}^{n+1}$.

\Rightarrow on an interval $|t - t_0| < h$, there is a unique solution to (*) defined on the interval and passing through p .

Note: The proof is similar to that of the 1-dim case.

Def (*) is linear if each F_i is linear in all of the x_i 's, $i = 1, \dots, n$. Indeed, if

$$(*) \begin{cases} x_1' = p_{11}(t)x_1 + \dots + p_{1n}(t)x_n + g_1(t) \\ x_2' = p_{21}(t)x_1 + \dots + p_{2n}(t)x_n + g_2(t) \\ \vdots \\ x_n' = p_{n1}(t)x_1 + \dots + p_{nn}(t)x_n + g_n(t) \end{cases}$$

and homogeneous if each $g_i(t) \equiv 0$.

Note: Solutions exist and are unique for a linear system of ODEs (like (*)) on an interval I if all $p_{ij}(t)$ and $g_i(t)$ are continuous on I .