110.302 Lecture 21: [redacted]

We have already seen that $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of the eigenvalue 3, for $A = \begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix}$.

What is an eigenvector for $\lambda = -2$?

Back to $A\vec{x} = \lambda \vec{x}$:

$$\begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

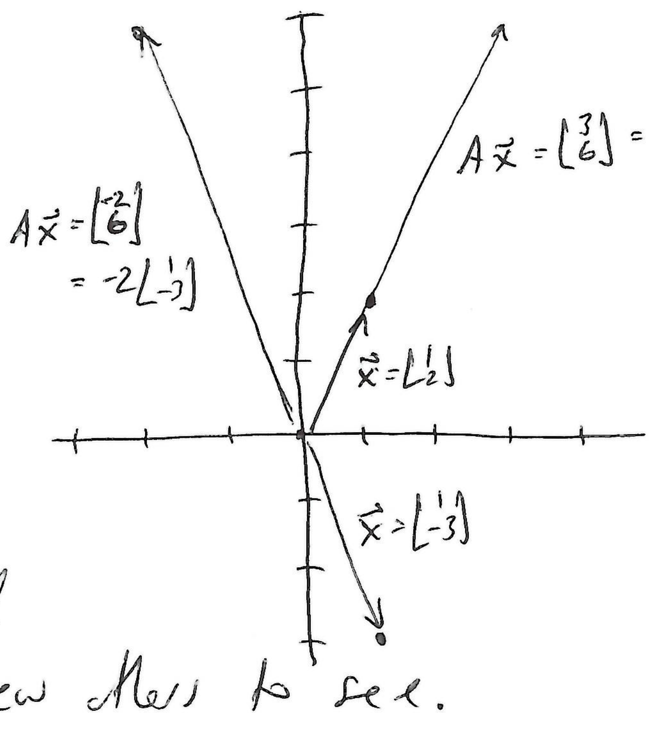
$$\left.\begin{aligned} x_1 + x_2 &= -2x_1 \\ 6x_1 &= -2x_2 \end{aligned}\right\} \Rightarrow \begin{aligned} 3x_1 &= -x_2 \\ 6x_1 &= -2x_2 \end{aligned}$$

Notice how this system is degenerate (has tons of solutions!) It <u>always is</u>...

Choose $x_1 = 1$ (any nonzero choice for x_1 will do)

$\Rightarrow x_2 = -3$, so

$\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ is an eigenvector for $\lambda = -2$. We say these 2 directions are invariant under A.



They are the only 2 such directions. Try a few others to see.

Back to ODEs:

If $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is a vector of variables (fncs of time)

then $\vec{x}\,' = \begin{bmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}$ is its derivative, and

$$x_1' = p_{11}(t)x_1 + \cdots + p_{1n}(t)x_n + g_1(t)$$

$$\vdots$$

$$x_n' = p_{n1}(t)x_1 + \cdots + p_{nn}(t)x_n + g_n(t)$$

is a linear system with matrix form

$$\vec{x}\,' = \mathbb{P}(t)\,\vec{x} + \vec{g}(t)$$

A solution is a vector of fncs $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$

and a set of solutions (if more than one) is

$$\vec{x}^{(1)}(t) = \begin{bmatrix} x_1^{(1)}(t) \\ \vdots \\ x_n^{(1)}(t) \end{bmatrix}, \ \ldots, \ \vec{x}^{(k)}(t) = \begin{pmatrix} x_1^{(k)}(t) \\ \vdots \\ x_n^{(k)}(t) \end{pmatrix}.$$

Note the notational confusion. We <u>will not deal</u> with nth order systems, so the context is ok.

Some facts - Let $\vec{x}' = \mathbb{P}(t)\vec{x}$ be a homogeneous linear system ($\vec{g(t)} = \vec{0}$).

① Superposition holds: if $\vec{x}^{(1)}(t)$ and $\vec{x}^{(2)}(t)$ both solve $\vec{x}' = \mathbb{P}(t)\vec{x}$, then so does

$$c_1\vec{x}^{(1)}(t) + c_2\vec{x}^{(2)}(t)$$

for any choice of $c_1, c_2 \in \mathbb{R}$.

(Any linear comb. of solutions is a solution!)

ex. $\vec{x}' = \begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix}\vec{x}$. Here $\mathbb{P}(t) = \begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix}$ are constants.

Verify that ① $\vec{x}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} = \begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix}$ is a solution.

$\dot{x}_1 = x_1 + x_2$

$\dot{x}_2 = 6x_1$

② $\vec{x}^{(2)} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t} = \begin{bmatrix} e^{-2t} \\ -3e^{-2t} \end{bmatrix}$ is also a solution.

Hence so is $c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t}$

For ②, $\vec{x}' = \frac{d}{dt}\begin{bmatrix} e^{-2t} \\ -3e^{-2t} \end{bmatrix} = \begin{bmatrix} e^{-2t} - 3e^{-2t} \\ 6e^{-2t} + 0(-3e^{-2t}) \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 6x_1 \end{bmatrix}$

(2) In $\mathbb{R}^n$, there can be at most n linearly independent vectors: Think

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots, \quad \vec{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

as an example.

It is reasonable to conclude that there can be up to n-linearly independent solutions to

$$\vec{x}\,' = \mathbb{P}(t)\,\vec{x}$$

$$\vec{x}^{(1)}(t) = \begin{bmatrix} x_1^{(1)}(t) \\ \vdots \\ x_n^{(1)}(t) \end{bmatrix}, \quad \dots, \quad \vec{x}^{(n)}(t) = \begin{bmatrix} x_1^{(n)}(t) \\ \vdots \\ x_n^{(n)}(t) \end{bmatrix}$$

They will be independent on some interval I if for $t \in I$,

$$c_1 \vec{x}^{(1)}(t) + \dots + c_n \vec{x}^{(n)}(t) = \vec{0}$$

can only be solved by $c_1 = \dots = c_n = 0$.

Combine all of these vector solutions as columns in a single $n \times n$ matrix:

$$\mathbf{X}(t) = \left[\vec{x}^{(1)} \dots \vec{x}^{(n)}\right] = \begin{bmatrix} x_1^{(1)} & \dots & x_1^{(n)} \\ \vdots & \ddots & \vdots \\ x_n^{(1)} & \dots & x_n^{(n)} \end{bmatrix}$$

Then $\det \mathbf{X} \neq 0$ iff all <u>columns are independent</u>

Def $\det \mathbb{X}(t)$ is called the Wronskian (determinant) of the solution set, and denoted $W(\vec{x}^{(1)}, \dots, \vec{x}^{(n)})$.

Thm If $\vec{x}^{(1)}, \dots, \vec{x}^{(n)}$ are all solutions to $\vec{x}' = \mathbb{P}(t)\vec{x}$ on $I = (\alpha, \beta) \in \mathbb{R}$, then for all $t \in \mathbb{I}$, either $W(\vec{x}^{(1)}, \dots, \vec{x}^{(n)})$ is ~~either~~ identically 0 or is never 0.

Def. If $\vec{x}^{(1)}, \dots, \vec{x}^{(n)}$ are all solutions to the n-dim $\vec{x}' = \mathbb{P}(t)\vec{x}$ on I, and $W(\vec{x}^{(1)}, \dots, \vec{x}^{(n)}) \neq 0$ on I

then (a) $\mathbb{X}(t)$ is called a fund. set of solutions

and (b) $\vec{\varphi}(t) = c_1\vec{x}^{(1)} + \dots + c_n\vec{x}^{(n)}$ is the general solution. $\vec{\varphi}(t) = \mathbb{X}(t)\vec{c}$

ex. Given $\vec{x}' = \begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix}\vec{x}$, with solutions

$$\vec{x}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} = \begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix}, \quad \vec{x}^{(2)} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t} = \begin{bmatrix} e^{-2t} \\ -3e^{-2t} \end{bmatrix}$$

Since
$$W(\vec{x}^{(1)}, \vec{x}^{(2)}) = \begin{vmatrix} e^{3t} & e^{-2t} \\ 2e^{3t} & -3e^{-2t} \end{vmatrix} = -3e^{3t}e^{-2t} - 2e^{3t}e^{-2t} = -5e^{t} \neq 0 \text{ on } \mathbb{R}$$

we have

$$\mathbb{X}(t) = \begin{bmatrix} e^{3t} & e^{-2t} \\ 2e^{3t} & -3e^{-2t} \end{bmatrix}$$

fundamental set of solutions

and

$$\vec{\varphi}(t) = c_1\begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix} + c_2\begin{bmatrix} e^{-2t} \\ -3e^{-2t} \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix}e^{3t} + c_2\begin{bmatrix} 1 \\ -3 \end{bmatrix}e^{-2t}$$

as the general solution