For \( \mathbf{x}' = \mathbf{A}_2 \mathbf{x} \).

\[ \mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{x}' = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \]

\[ \mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{x}' = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]

\[ \mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{x}' = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

One can construct a slope field in \( \mathbb{R}^2 \) via matrix multiplication: For \( \mathbf{x} \in \mathbb{R}^2 \),

\[ \mathbf{x}' = \mathbf{A}_2 \mathbf{x} \]

the tangent to the solution curve passing through \( \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \mathbf{x} \).

The tangent to the solution curve passing through \( \mathbf{x} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \), \( \mathbf{x}' = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \)

is helpful here.

Solutions curves are integral curves of the slope field:

Given \( c_1, c_2 \in \mathbb{R} \), the curve

\[ \mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t} \]

is one of these curves.
6. Straight line motion only occurs when one of \( c_1, c_2 \) is 0.

Let \( c_1 = -2, c_2 = 0 \).

\[ x(t) = -2L_1 e^t \]

\[ = [-2] e^t \]

7. A copy of \( \mathbb{R}^2 \) with enough representative curve on it to give a good sense of solutions is called a phase portrait.

8. Solutions are called trajectories or orbits.

9. General long term behavior of trajectories can be read off easily from a phase portrait.

Given \( x(t) = c_1 L_1 e^{3t} + c_2 L_2 e^{-2t} \).
\( \Pi \circ \omega \) cont'd.

- \( 16 \ C_1 = 0, \ C_2 \neq 0 \), hence \( x(t) = c_2 e^{-at} \)

and

\[
\lim_{t \to \infty} x(t) = \left\lfloor 0 \right\rfloor.
\]

How about \( \lim_{t \to -\infty} x(t) \)?

We say it is \underline{unbounded}.

Note: Solutions never touch nor cross lines.

So since \( x(t) = \left\lfloor 0 \right\rfloor \) is an equilibrium soln,
no other solution actually reaches \( \left\lfloor 0 \right\rfloor \).

- \( 16 \ C_1 \neq 0, \ C_2 = 0 \)?

\[
\lim_{t \to \infty} x(t) \text{ does not exist or } x(t) \text{ is unbounded}.
\]

\[
\lim_{t \to -\infty} x(t) = \left\lfloor 0 \right\rfloor.
\]

- \( 16 \ C_1 \neq 0, \ C_2 \neq 0 \)? These are the curved lines in the portrait. For these, what can we say about the forward trajectory \( \lim_{t \to \infty} x(t) \), or the backward one \( \lim_{t \to -\infty} x(t) \)?
One answer? Given \( \mathbf{x}(t) = c_1 [\frac{1}{2}] e^{3t} + c_2 [\frac{1}{3}] e^{-2t} \) and \( c_1 \neq 0, c_2 \neq 0 \), then as \( t \) gets large \( (t \to \infty) \), \( \mathbf{x}(t) \) looks more and more like \( c_1 [\frac{1}{2}] e^{3t} \), and less and less like \( c_2 [\frac{1}{3}] e^{-2t} \).

How about in backward time?

\[ \text{III Back to solution building via the properties of } A \]

\( A \): For \( \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \),

\[ \mathbf{x}(t) = c_1 [\frac{1}{2}] e^{3t} + c_2 [\frac{1}{3}] e^{-2t} \]

the eigenvalues of \( A \) (\( \lambda_1 = 3, \lambda_2 = -2 \)), and representative eigenvectors \( (\mathbf{v}_1 = [\frac{1}{2}], \mathbf{v}_2 = [\frac{1}{3}]) \) are explicitly not of the solution. Why?
Here, the eigenvalues and eigenvectors of a matrix \( A \) satisfy \( A \vec{v} = \lambda \vec{v} \), or
\[(A - \lambda I) \vec{v} = \vec{0}.

For this to have nontrivial solutions for \( \vec{v} \),
- \( \lambda \) must be an eigenvalue, and
- \( \det (A - \lambda I) \vec{v} = \vec{0} \).

For \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \), \( \det (A - \lambda I) = \begin{vmatrix} \lambda - a & b \\ c & \lambda - d \end{vmatrix} = 0 \)
\[= \lambda^2 - (a+d) \lambda + (ad-bc) = 0 \]

This is called the characteristic \underline{equation} of \( A \), and solutions can be
- \( \lambda \) real, distinct
- \( \lambda \) real and repeated
- complex conjugates.
How to relate to solutions of $\dot{x} = Ax$?

- Recall $x = Ax$ is solved by $x(t) = e^{At}$
- Assume $\dot{x} = Ax$ is also solved by exponential's. For $n=2$, $x = [x_1 \, x_2]'$

Assume $x_1(t) = c_1 e^{rt}$, $x_2(t) = c_2 e^{rt}$

$$\Rightarrow \dot{x}(t) = [c_1] e^{rt}, \text{ and } \dot{x}' = r [c_1] e^{rt}$$

and hence $\dot{x}' = Ax$ is $r [c_1] e^{rt} = A [c_1] e^{rt}$

or $r [c_1] = A [c_1]$, i.e. $r \vec{V} = A \vec{V}$.

Add what do solutions to this look like?!

- $r$ is an eigenvalue
- $\vec{V}$ is an eigenvector of $A$.

Notes

1. This works exactly well for $n=2$.
2. In the case where $A_{nn}$ is real and symmetric (i.e. when $a_{ii} = a_{ji}$)

   $\Rightarrow$ all eigenvalues are real and even if repeated, there is a full set of orthogonal basis.
We have the following:

For $\hat{\mathbf{x}} = A_{\text{eig}} \mathbf{x}$, when all eigenvalues of $A$ are real and distinct, then

$$\hat{\mathbf{x}}(t) = c_1 \hat{\mathbf{v}}_1 e^{\lambda_1 t} + \cdots + c_n \hat{\mathbf{v}}_n e^{\lambda_n t}$$

is the general soln, where $\hat{\mathbf{v}}_i$ is the corresponding eigenvector of $\mathbf{v}_i$ for $A$. 