110.302 Lecture 24

Back to $\vec{x}' = A_{2\times2}\vec{x}$ in the case where the 2 eigenvalues of A, r_1, r_2 are real and distinct, so $r_1 \neq r_2$.

Then the general solution is

$$\vec{x}(t) = c_1 \vec{v}_1 e^{r_1 t} + c_2 \vec{v}_2 e^{r_2 t}$$

where $(r_1, \vec{v}_1)$ and $(r_2, \vec{v}_2)$ are eigenvalue/eigenvector pairs.

<u>Some Notes</u> ① Works fine for 0 as an eigenvalue:

ex. $\vec{x}' = \begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix} \vec{x}$, where $r_1 = 0$, $r_2 = 2$,

$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and

$$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{0t} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{2t}$$

$$= c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{2t}$$

(2) Still works for repeated eigenvalues only when there are enough eigenvectors.

ex. $\vec{x}' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{x}$. Char eqn is $r^2 - 2r + 1 = 0$

solved by $r_1 = r_2 = 1$.

The eigenvector eqn is then $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$.

$\Longrightarrow \left.\begin{matrix} v_1 = v_1 \\ v_2 = v_2 \end{matrix}\right\}$ can be solved by more than 1 lin. ind. eigenvector.

One choice of eigenvectors is $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

So soln is $\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t$

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ex.

$$\vec{x}' = \underbrace{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}}_{A} \vec{x}.$$

Here, $\det(A - rI_3) = \begin{vmatrix} -r & 1 & 1 \\ 1 & -r & 1 \\ 1 & 1 & -r \end{vmatrix}$

$= -r \begin{vmatrix} -r & 1 \\ 1 & -r \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 1 & -r \end{vmatrix} + 1 \begin{vmatrix} 1 & -r \\ 1 & 1 \end{vmatrix}$

$= -r(r^2 - 1) - 1(-r - 1) + 1(1 + r) = 0$

$= r^3 - 3r + 2 = 0$

Long division yields

$$r^3 - 3r + 2 = (r+1)(r^2 - r - 2) = (r+1)^2(r-2)$$

So $r_1 = 2$, $r_2 = r_3 = -1$.

Book calculated eigenvectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

So solution is

$$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{-t}$$

exercise: Imagine a 3-d phase space....
What happens to solutions near the ~~fate~~ equilibrium at the origin?

(3) This formulation for finding solutions <u>does not</u> work when there are not enough independent eigenvectors for repeated eigenvalues.

ex. $\vec{x}' = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} \vec{x}$. Here, eigenvalues $r_1 = r_2 = -2$.

But the eigenvector eqn $A\vec{v} = -2\vec{v}$, or

$$-2x_1 + x_2 = -2x_1$$
$$-2x_2 = -2x_2$$

is <u>ONLY</u> solved by $x_2 = 0$, $x_2 \neq 0$. An only independent choice would be $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Here, we will need to come up with another method to find another independent soln.

Properties of phase portraits

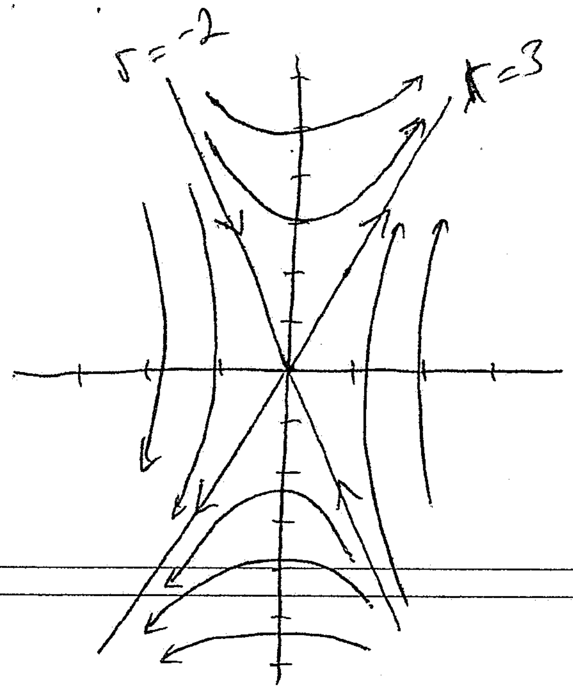
Let $\vec{x}' = A\vec{x}$. We have

① For *any* $A_{2\times 2}$, the origin is an equilibrium solution ($\vec{x} = \begin{bmatrix}0\\0\end{bmatrix}$ is always a solution to $A\vec{x} = \vec{0}$).

② If $\det A \neq 0$, then $\vec{x}(t) = \begin{bmatrix}0\\0\end{bmatrix}$ is the ONLY equilibrium solution (stationary pt, or fixed pt), since $A\vec{x} = \vec{0}$ has $\begin{bmatrix}0\\0\end{bmatrix}$ as the only solution.

③ The eigenvectors of A correspond to lines through the origin where the solutions exhibit straight line motion

Note: These straight lines contain many distinct solutions.



phase portrait for
$\vec{x}' = \begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix} \vec{x}$.
w/ solns
$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t}$

(4) The sign of each eigenvalue determines direction along lines (toward origin if <0 outward if >0).

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(5) If eigenvalues are real and distinct then these are the only straight lines (why?)

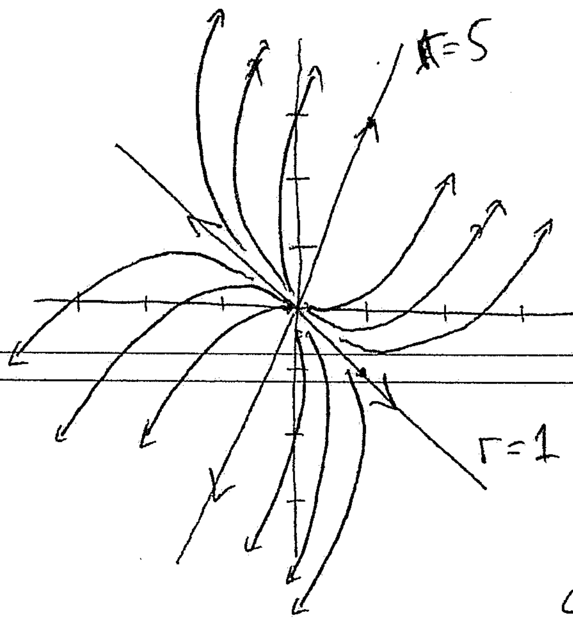
(6) Signs of eigenvalues determine "stability" of equilibrium at origin

(How solutions behave "near" the origin. Do they stay nearby, converge to, or diverge from....)

(7) Above, origin is called a "saddle pt".
Would you consider it stable?

ex. $\vec{x}' = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$. Here $r_1 = 5$, $r_2 = 1$,

$\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.



The general solution is

$$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{t}$$

All solutions tend to origin as $t \to -\infty$ and are unbounded as $t \to \infty$

Origin is called a source and is unstable.

Q: Why is motion curved like above? How to determine?

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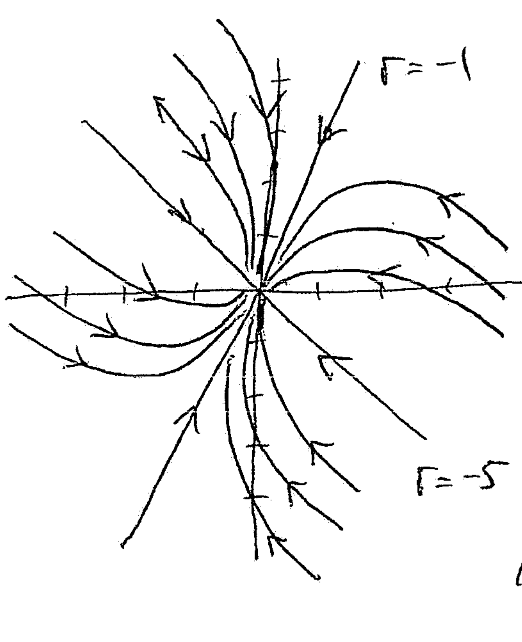
ex. $\vec{x}' = \begin{bmatrix} -4 & 1 \\ 3 & -2 \end{bmatrix} \vec{x}$ Here $r_1 = -5$, $r_2 = -1$, with

$\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$



General solution is

$$\vec{x}(t) = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-5t} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t}$$

Phase portrait is similar to above but different: how?

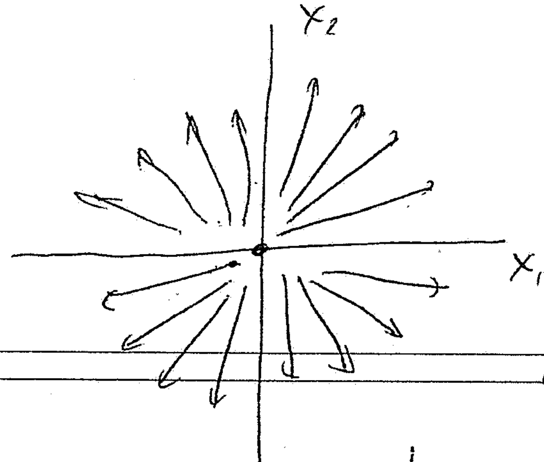
Here origin is a "sink" and asymptotically stable.

ex. $\vec{x}' = \begin{bmatrix}1&0\\0&1\end{bmatrix}\vec{x}$. Here, as we have seen

$r_1 = r_2 = 1$, $\vec{v}_1 = \begin{bmatrix}1\\0\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}0\\1\end{bmatrix}$

and $\vec{x}(t) = c_1\begin{bmatrix}1\\0\end{bmatrix}e^t + c_2\begin{bmatrix}0\\1\end{bmatrix}e^t$

$= \left(c_1\begin{bmatrix}1\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\end{bmatrix}\right)e^t$



This is another source at the origin here called a <u>star node</u>.

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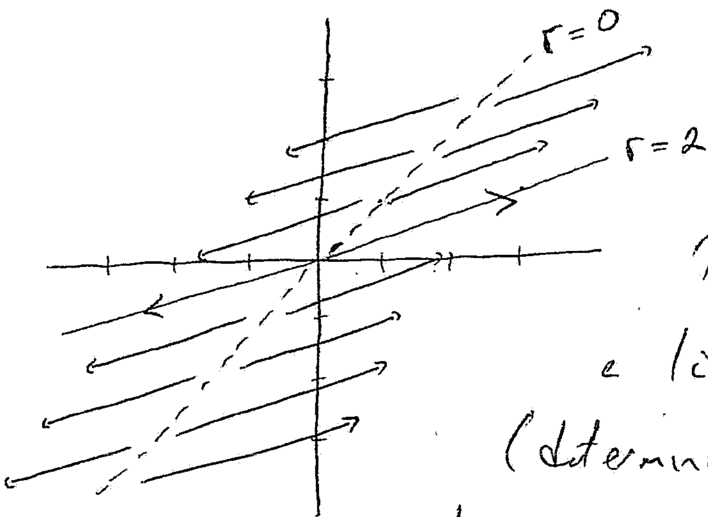
ex. $\vec{x}' = \begin{bmatrix}3&-3\\1&-1\end{bmatrix}\vec{x}$. Here $r_1 = 0$, $r_2 = 2$,

$\vec{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}3\\1\end{bmatrix}$.

General soln is

$\vec{x}(t) = c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}3\\1\end{bmatrix}e^{2t}$



This one is special: there is a line of equilibrium solution (determinant is 0). Off the dotted line, motion is straight along lines parallel to $\begin{bmatrix}3\\1\end{bmatrix}$ and moving out from dotted line.

Q: What is the stability?