110.302 Lecture 25:

New question: What if the eigenvalues (solns to the characteristic eqn of $A_{2\times 2}$ in $\vec{x}' = A\vec{x}$ are not real?

Then they are complex (the discriminant b^2-4ac of the quadratic formula used to solve the char. eqn is < 0).

They must be complex conjugates (why?)

How to use them?

Let's play the same game for constructing solutions to $\vec{x}' = A\vec{x}$ using eigenvalue eigenvector pairs:

For $\vec{x}' = A_{2\times 2}\vec{x}$, suppose $r_1 \neq r_2$ are two distinct solns to char. eqn of A and we calculate eigenvectors $\vec{v}_1, \vec{v}_2$ respectively to r_1, r_2. Then general soln is

$$\vec{x}(t) = c_1\vec{v}_1 e^{r_1 t} + c_2\vec{v}_2 e^{r_2 t}$$

We try this with complex r:

ex. $\vec{x}' = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}\vec{x}$. char. eqn is $r^2+2r+2=0$, solved by $r=-1\pm i$

Leave them as $r_1=-1+i$, $r_2=-1-i$ and solve for $\vec{v}_1$: $\quad A\vec{v}=r\vec{v}$

$$\begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = (-1+i)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \text{ or}$$

$$v_2 = -v_1 + iv_1$$

$$-2v_1 - 2v_2 = -v_2 + iv_2$$

We can substitute (1) into (2) and simplify to get $-2i\,v_1 = -2i\,v_1$, solved by any choice of v. For example, choose $v_1 = 1$. Then $v_2 = (-1+i)$ and

$$r_1 = -1+i, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ -1+i \end{bmatrix}$$

forms an "eigenvalue/eigenvector" pair

<u>Notes</u> ① This is not quite accurate since the definition of eigenvector is that of a vector whose direction does not change upon mult by a matrix. Without real eigenvalues, there are no real eigenvectors! But the term "complex eigenvector" is a commonly used one.

Notes cont'd.

(2) The other eigenvalue/eigenvector pair is

$r_2 = -1 - i, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1-i \end{bmatrix}$. (check this!)

(3) Rewrite $\vec{v}_1 = \begin{bmatrix} 1 \\ -1+i \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} i = \vec{a} + i\vec{b}$

Then along with $r_1 = -1 + i = \lambda + i\mu$

we can attempt to form solutions.

General idea for a Method for constructing solutions?

- Given $r_1 = \lambda + i\mu, \quad \vec{v}_1 = \vec{a} + i\vec{b}$

$r_2 = \lambda - i\mu, \quad \vec{v}_2 = \vec{a} - i\vec{b}$,

Create a "complex" solution in the normal way:

$$\vec{x}(t) = \vec{v}_1 e^{r_1 t} = (\vec{a} + i\vec{b}) \underbrace{e^{\lambda t}(\cos\mu t + i\sin\mu t)}_{\text{Euler formula for } e^{r_1 t}}$$

$$= e^{\lambda t}(\vec{a}\cos\mu t - \vec{b}\sin\mu t)$$
$$+ i e^{\lambda t}(\vec{a}\sin\mu t + \vec{b}\cos t)$$

Hence we can write $\vec{x}(t) = \vec{u}(t) + i\,\vec{w}(t)$, where

$$\vec{u}(t) = e^{\lambda t}(\vec{a}\cos\mu t - \vec{b}\sin\mu t)$$
$$\vec{w}(t) = e^{\lambda t}(\vec{a}\sin\mu t + \vec{b}\cos\mu t)$$

Notes ① These are 2 real-valued functions which each solve the ODE $\vec{x}' = A\vec{x}$ (check this!)

② They are independent (check the Wronskian).

③ The general solution to $\vec{x}' = A\vec{x}$ when eigenvalues $r = \lambda \pm i\mu$ are complex and with eigenvectors $\vec{v} = \vec{a} \pm i\vec{b}$ is

$$\vec{x}(t) = c_1\vec{u}(t) + c_2\vec{w}(t)$$

Back to example: $\vec{x}' = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}\vec{x}$.

Here $r_1 = \lambda + i\mu = -1 + i$, $\vec{v}_1 = \vec{a} + i\vec{b} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + i\begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

so

$$\vec{x}(t) = c_1 e^{-t}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\cos t - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\sin t\right) + c_2 e^{-t}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\sin t + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\cos t\right)$$