110.302 Lecture 26: $\overline{\underline{\text{VI}}}$

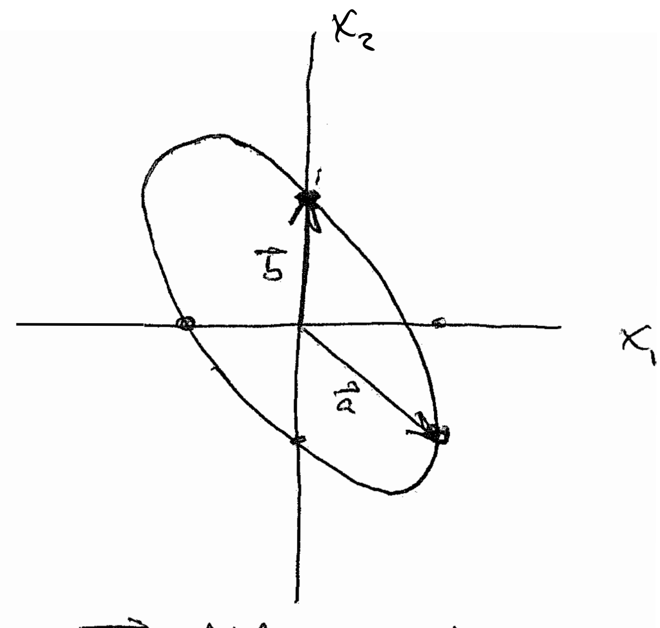
Q1: What do solutions look like in the phase plane when characteristic eqn has no real solutions for $\vec{x}' = A_{2\times2}\vec{x}$?

Q2: What does the parameterized curve in $\mathbb{R}^2$,

$$\vec{f(t)} = \vec{a}\cos\mu t - \vec{b}\sin\mu t$$

look like?

- periodic w/ period $\frac{2\pi}{\mu}$.
- for any $t \in \mathbb{R}$, curve pt will always be a linear combo. of $\vec{a}, \vec{b}$.



$$\vec{f(t)} = \begin{bmatrix}1\\-1\end{bmatrix}\cos t - \begin{bmatrix}0\\1\end{bmatrix}\sin t$$

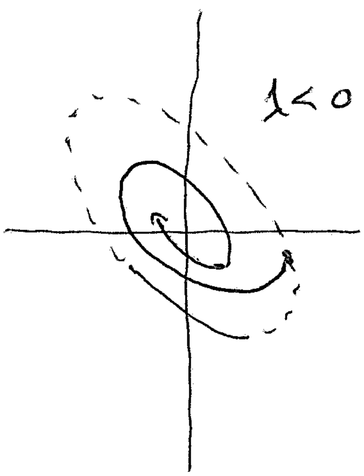
exercise: How does one locate the major axis of this ellipse?

- Image will be an ellipse, with major axis "near" (but definitely not on, in general) to $\vec{a}$.

Q3: So what do solutions $\vec{x}(t)$ of $\vec{x}' = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \vec{x}$ look like?

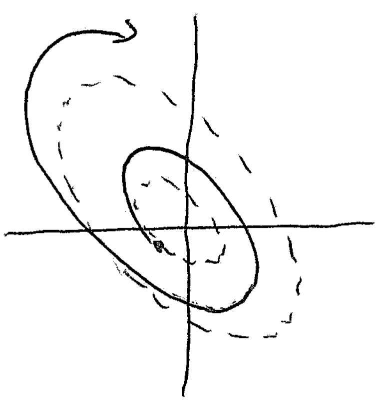
A3: First, what is the effect of λ in $r = \lambda + i\mu$?

- if $\lambda = 0$, $r = i\mu$ is purely imaginary; all solutions are ellipses encircling the origin.



- if $\lambda < 0$, as $\overrightarrow{f(t)}$ traverses one period, $\vec{x}(t)$ changes its magnitude by a factor of $e^{\lambda t} < 1$.

Trajectories spiral inward....



- if $\lambda > 0$, spiral outward.
- direction of travel?

① Given $\overrightarrow{f(t)}$, calculate $\vec{f}'(t)$

② Evaluate $\vec{x}(0)$ and $\vec{x}(t)$ where $t = \left(\frac{2\pi}{\mu}\right)\frac{1}{4}$ ($\frac{1}{4}$ turn around the ellipse).

Back to stability:

In this case where $r_1 = \lambda + i\mu$, $r_2 = \lambda - i\mu$ as long as $\mu \neq 0$, $r_1 \neq r_2$ ~~difference~~ $\neq 0$. Thus the origin is the ONLY equilibrium. What is its stability?

- $\lambda < 0$: all solutions tend toward origin
- $\lambda > 0$: all solutions diverge from origin (tend toward origin as $t \to -\infty$).
- $\lambda = 0$: Solutions are bounded and neither tend toward nor away from origin.

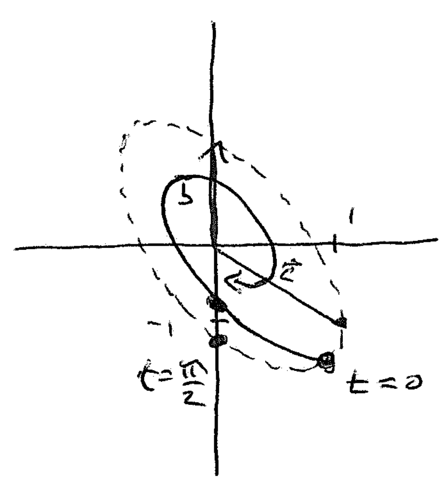
ex. $\vec{x}' = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \vec{x}$. Solution w/ ~~$c_1, c_2$~~, $c_1 = 1$, $c_2 = 0$ is

$$\vec{x}(t) = e^{-t}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos t - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin t\right)$$

@ $t = 0$, $\vec{x}(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

@ $t = \frac{\pi}{2}$, $\vec{x}(t) = e^{-\pi/2}\begin{bmatrix} 0 \\ -1 \end{bmatrix}$

Here the spiral is inward and clockwise.



For $\vec{x}' = \mathbb{P}(t)\vec{x}$

<u>Def.</u> An equilibrium solution $\vec{p}$ is called <u>asymptotically</u> <u>stable</u> if $\exists\, \varepsilon > 0$ such that for all solutions $\vec{x}(t)$, with $\vec{x}(t_0) = \vec{x}^{\circ}$, we have

$$\text{if } \|\vec{x}^{\circ} - \vec{p}\| < \varepsilon, \text{ then } \lim_{t \to \infty} \vec{x}(t) = \vec{p}.$$

<u>Notes</u> ① Anything that starts within ε of $\vec{p}$ is asymptotic to $\vec{p}$ as $t \to \infty$.

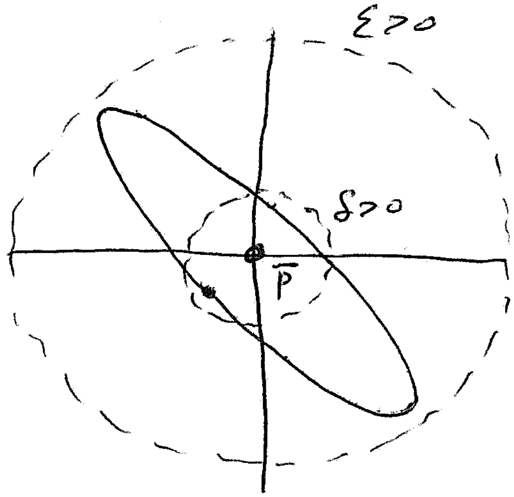
② For a linear system, if $\exists\, \varepsilon > 0$ that works then <u>any</u> $\varepsilon > 0$ will work (why?).

③ An asymptotically stable equilibrium is called a <u>sink</u>

<u>Def</u> An equilibrium soln $\vec{p}$ is called <u>stable</u> if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all solutions $\vec{x}(t)$, where $\vec{x}(t_0) = \vec{x}^{\circ}$, we have

$$\text{if } \|\vec{x}^{\circ} - \vec{p}\| < \delta, \text{ then } \|\vec{x}(t) - \vec{p}\| < \varepsilon$$

$\forall\, t \geq t_0$.



Note: ① Given an ε-nbhd of $\bar p$, if you can always find a smaller nbhd (a δ-nbhd) of $\bar p$ so that if you start in the δ-nbhd you never leave the ε-nbhd,

then $\bar p$ is stable

② Any $\bar p$ which is asymptotically stable is also stable. But only sinks are asympt. stable.

③ An unstable equilibrium which is backward asymptotically stable (as $t\to-\infty$) is called a source

④ For $\vec x' = A_{2\times 2}\vec x$, with $r_1=\lambda+i\mu$, $r_2=\lambda-i\mu$, $\mu\neq 0$, if $\lambda=0$, (so that trajectories are ellipses), then $\vec 0$ is called a center