110.302 Lecture 27:

Recall our fundamental set of solutions to

$$\vec{x}' = \begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix} \vec{x} \quad \text{is} \quad \Psi(t) = \begin{bmatrix} e^{3t} & e^{-2t} \\ 2e^{3t} & -3e^{-2t} \end{bmatrix}$$

where the columns of $\Psi(t)$ correspond to the 2 independent solutions

$$\vec{x}^{(1)}(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}, \quad \vec{x}^{(2)}(t) = \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t}$$

They are independent, since $\det \Psi(t) \neq 0$.

The general solution is then

$$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{3t} + c_2 e^{-2t} \\ 2c_1 e^{3t} - 3c_2 e^{-2t} \end{bmatrix} = \begin{bmatrix} e^{3t} & e^{-2t} \\ 2e^{3t} & -3e^{-2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\vec{x}(t) = \Psi(t)\vec{c}, \quad \text{where} \quad \vec{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Here, given any initial data $\vec{x}(t_0) = \vec{x}^0 = \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix}$

we have to solve $\vec{x}^0 = \Psi(t_0)\vec{c}$ for $\vec{c}$ ....

Since $\det \Psi(t) \neq 0$, we have $\vec{c} = \Psi^{-1}(t_0)\vec{x}^0$.

With this, we can change the form of our general solution to directly reflect the initial data (instead of the constants $\vec{c}$)

$$\vec{x}(t) = \mathbf{X}(t)\vec{c} = \mathbf{X}(t)\left(\mathbf{X}^{-1}(t_0)\,\vec{x}^{\,0}\right)$$
$$= \underbrace{\left(\mathbf{X}(t)\,\mathbf{X}^{-1}(t_0)\right)}\vec{x}^{\,0}$$
$$= \Phi(t)\,\vec{x}^{\,0}$$

where $\Phi(t) = \mathbf{X}(t)\,\mathbf{X}^{-1}(t_0)$ is simply another choice of fundamental ~~set~~ of solutions, but one that has some special properties.

<u>Notes</u> ① Useful since initial data coincide with constants of integration

② Easy to calculate if $\mathbf{X}(t)$ is known: $\mathbf{X}^{-1}(t_0)$ is still a 2x2 matrix of numbers following $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

③ Works in general for <u>any</u> homogeneous IVP $\vec{x}' = \mathbf{P}(t)\,\vec{x}$, $\vec{x}(t_0) = \vec{x}^{\,0}$.

(4) Nothing is gained when solving a single IVP. But if you need to solve many?

ex Solve $\vec{x}' = \begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix} \vec{x}$, $\vec{x}(0) = \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \vec{x}^{\circ}$

Solution: Here $\mathbf{X}(t) = \begin{bmatrix} e^{3t} & e^{-2t} \\ 2e^{3t} & -3e^{-2t} \end{bmatrix}$, $\mathbf{X}(0) = \begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix}$

since $t_0 = 0$. The general solution is

$\vec{x}(t) = \mathbf{X}(t)\,\vec{c}$, where $\vec{c}$ is given by

$\vec{x}(0) = \mathbf{X}(0)\,\vec{c} \implies \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$\implies \left.\begin{aligned} 4 &= c_1 + c_2 \\ 0 &= 2c_1 - 3c_2 \end{aligned}\right\} c_1 = \frac{12}{5},\ c_2 = \frac{8}{5}$

So $\vec{x}(t) = \begin{bmatrix} e^{3t} & e^{-2t} \\ 2e^{3t} & -3e^{-2t} \end{bmatrix} \begin{bmatrix} 12/5 \\ 8/5 \end{bmatrix}$

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ex. Solve $\vec{x}' = \begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix} \vec{x}$, $\vec{x}(0) = \begin{bmatrix} m \\ 0 \end{bmatrix}$, $m = 1, 2, \cdots, 1000$

Note: The above procedure, with system

$m = c_1 + c_2$
$0 = 2c_1 - 3c_2$

would have to be repeated 1000 times!

Or, first switch to $\Phi(t)$ as the fund. set of solns.

ex. Solve $\vec{x}\,' = \begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix} \vec{x}$, $\vec{x}_m(0) = \begin{bmatrix} m \\ 0 \end{bmatrix}$, $m = 1, \cdots, 1000$

Solution: Again $X(t) = \begin{bmatrix} e^{3t} & e^{-2t} \\ 2e^{3t} & -3e^{-2t} \end{bmatrix}$, $X(0) = \begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix}$

Here $X^{-1}(0) = \frac{1}{\det X(0)} \begin{bmatrix} -3 & -1 \\ -2 & 1 \end{bmatrix} = \frac{1}{-5} \begin{bmatrix} -3 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 3/5 & 1/5 \\ 2/5 & -1/5 \end{bmatrix}$

So $X(t) X^{-1}(0) = \begin{bmatrix} e^{3t} & e^{-2t} \\ 2e^{3t} & -3e^{-2t} \end{bmatrix} \begin{bmatrix} 3/5 & 1/5 \\ 2/5 & -1/5 \end{bmatrix}$

$$= \begin{bmatrix} \frac{3}{5}e^{3t} + \frac{2}{5}e^{-2t} & \frac{1}{5}e^{3t} - \frac{1}{5}e^{-2t} \\ \frac{6}{5}e^{3t} - \frac{6}{5}e^{-2t} & \frac{2}{5}e^{3t} + \frac{3}{5}e^{-2t} \end{bmatrix}$$

And for each $\vec{x}_m^{\,0} = \begin{bmatrix} m \\ 0 \end{bmatrix}$, we have

$$\vec{X}_m(t) = \begin{bmatrix} \frac{3}{5}e^{3t} + \frac{2}{5}e^{-2t} & \frac{1}{5}e^{3t} - \frac{1}{5}e^{-2t} \\ \frac{6}{5}e^{3t} - \frac{6}{5}e^{-2t} & \frac{2}{5}e^{3t} + \frac{3}{5}e^{-2t} \end{bmatrix} \begin{bmatrix} m \\ 0 \end{bmatrix}$$

$\blacksquare$

Notes ① The fundamental set of solutions

$\Phi(t) = \Psi(t)\,\Psi^{-1}(t_0)$ is special

since $\Phi(0) = I$ (check this in last example).

② Any choice of $\Psi(t)$ relates the initial data (in x_1x_2-coords) to the unknown constants $\vec{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ (in coordinates given by eigenvectors).

By choosing $\Phi(t)$ as your fund. set of solns, you are choosing the coord. system so that $\vec{x}^0 = \vec{c}$.

③ Any fundamental matrix $\Psi(t)$ of $\vec{x}' = \mathbb{P}(t)\vec{x}$ solves the "matrix" form of the ODE:

$$\Psi'(t) = \mathbb{P}(t)\,\Psi(t)$$

where $\Phi'(t) = \frac{d}{dt}\Phi(t)$ (each entry is ~~derived~~ differentiated)

By "choosing" $\Phi(t)$, so that $\Phi'(t) = A\Phi(t)$, solutions are "exponential," and we can write $\Phi(t) = e^{At}$, A a matrix.

This only makes sense in its Taylor Expansion

$$e^{At} = \sum_{n=0}^{\infty} \frac{A^n}{n!} t^n = I + At + \frac{1}{2}A^2t^2 + \frac{1}{6}A^3t^3 + \cdots$$

This is a very important exponential but is a bit too deep for this class.