110.302 Lecture 28:

Now consider the system

$$\dot{x}_1 = -2x_1 + x_2 \qquad\qquad \text{or} \qquad \vec{x}\,' = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} \vec{x}$$
$$\dot{x}_2 = \qquad -2x_2$$

Here eigenvalues of $A = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}$ satisfy $r^2 + 4r + 4 = 0$

or $r_1 = -2 = r_2$.

Eigenvectors satisfy $A\vec{v} = -2\vec{v}$:

$$\left.\begin{aligned} -2v_1 + v_2 &= -2v_1 \\ -2v_2 &= -2v_2 \end{aligned}\right\} \quad \begin{aligned} v_2 &= 0 \\ v_1 &= \text{anything} \end{aligned}$$

All eigenvectors look like $\begin{bmatrix} * \\ 0 \end{bmatrix}$. Hence there is only 1 lin. indep. eigenvector. Choose $* = 1$.

For $r = -2$, $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. and 1 soln is $\vec{x}^{(1)}(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-2t}$.

How to find another one?

Q: What did we do for the case of a 2^{nd} order lin. homogeneous ODE with constant coefficients when there was only one root to the char. eqn?

A: Two indep solutions were e^{-2t} and te^{-2t}.

Lets try this here:

Create a guess for a solution:

$$\vec{x}^{(2)}(t) = \vec{w}\, t e^{-2t} \qquad \text{for } \vec{w} \text{ a 2-vector}$$

For this to be a solution, it must "fit" the ODE: $\vec{x}' = A\vec{x}$

For this example, we get

$$\frac{d}{dt}\left[\vec{w}te^{-2t}\right] = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}(\vec{w}te^{-2t})$$

$$\vec{w}e^{-2t} - 2\vec{w}te^{-2t} = A\vec{w}te^{-2t}$$

$$\vec{w} - 2\vec{w}t = A\vec{w}t$$

exercise: Show there are no nontrivial ~~vect~~ constant vectors $\vec{w}$ that work here for all $t \in \mathbb{R}$.

So we try again: Create an idea of what the general solution should look like and see if we can find $\vec{w}$:

Let $\vec{x}^{(2)}(t) = \vec{v}te^{-2t} + \vec{w}e^{-2t}$, and try to solve for both $\vec{v}$ and $\vec{w}$ at the same time.

We get again:

$$\frac{d}{dt}\left[\vec{v}te^{-2t} + \vec{w}e^{-2t}\right] = A(\vec{v}te^{-2t} + \vec{w}e^{-2t})$$

$$\vec{v}e^{-2t} - 2\vec{v}te^{-2t} - 2\vec{w}e^{-2t} = A\vec{v}te^{-2t} + A\vec{w}e^{-2t}$$

$$\vec{v} - 2\vec{w} - 2\vec{v}t = A\vec{v}t + A\vec{w}$$

These 2 sides are polynomials in t:

The coefficients in t: $\quad -2\vec{v} = A\vec{v}$

This is solved precisely when $\vec{v}$ is an eigenvector: $\vec{v} = \begin{bmatrix}1\\0\end{bmatrix}$.

The constant coefficients: $\vec{v} - 2\vec{w} = A\vec{w}$

Rearrange to get $(A - (-2I))\vec{w} = \vec{v}$.

In this last equation, we call $\vec{w}$ a generalized eigenvector: $\left(\begin{bmatrix}-2 & 1\\0 & -2\end{bmatrix} - \begin{bmatrix}-2 & 0\\0 & -2\end{bmatrix}\right)\begin{bmatrix}w_1\\w_2\end{bmatrix} = \begin{bmatrix}v_1\\v_2\end{bmatrix}$

$$\Longrightarrow \quad \left.\begin{array}{l} 0w_1 + 1w_2 = v_1 \\ 0w_1 + 0w_2 = v_2 \end{array}\right\} \quad w_2 = 1 \quad w_1 = \text{anything.}$$

choose $\vec{w} = \begin{bmatrix}0\\1\end{bmatrix}$. Then our guess

$$\vec{x}^{(2)}(t) = \vec{v}te^{-2t} + \vec{w}e^{-2t} = \begin{bmatrix}1\\0\end{bmatrix}te^{-2t} + \begin{bmatrix}0\\1\end{bmatrix}e^{-2t}$$

is another solution to $\vec{x}' = \begin{bmatrix}-2 & 1\\0 & -2\end{bmatrix}\vec{x}$.

Q1: Is it a solution? Yes because we constructed it.

Q2: Is it independent of the other one?

$$\vec{x}^{(1)}(t) = \begin{bmatrix}e^{-2t}\\0\end{bmatrix}, \quad \vec{x}^{(2)}(t) = \begin{bmatrix}te^{-2t}\\e^{-2t}\end{bmatrix}$$

$$W(\vec{x}^{(1)}, \vec{x}^{(2)}) = \begin{vmatrix}e^{-2t} & te^{-2t}\\0 & e^{-2t}\end{vmatrix} = e^{-4t} \neq 0 \text{ on } \mathbb{R}.$$

Yes, independent.

Here our general solution is

$$\vec{x}(t) = c_1 \begin{bmatrix}1\\0\end{bmatrix} e^{-2t} + c_2\left(\begin{bmatrix}1\\0\end{bmatrix} t e^{-2t} + \begin{bmatrix}0\\1\end{bmatrix} e^{-2t}\right)$$

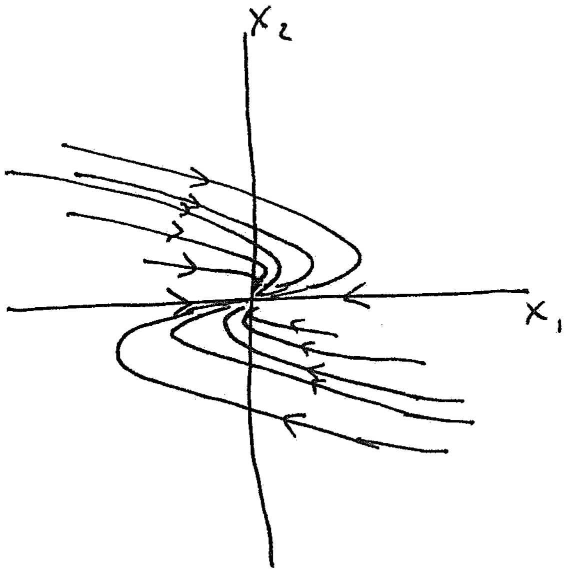
<u>To recap</u>: Given $\vec{x}' = A\vec{x}$, where A has a repeated eigenvalue r and only 1 linearly indep. eigenvector $\vec{v}$, then the general solution is

$$\vec{x}(t) = c_1 \vec{v} e^{rt} + c_2\left(\vec{v} t e^{rt} + \vec{w} e^{rt}\right)$$

where $\vec{w}$ solves $(A - rI_2)\vec{w} = \vec{v}$.

What does this solution look like?

- Only straight line motion is along single eigenvector line.



- Other vector $\vec{w}$ helps to fill out the other dimension, but with the there is no motion along it that is straight
- Called a degenerate node, stable when $r < 0$ unstable when $r > 0$.

or an <u>improper node</u>.