In preparation for the study of non-linear systems, Section 9.1 is a review of linear systems theory, but given solely in terms of understanding how to characterize the type and stability of the equilibrium solution at the origin of the phase plane.

**General Facts**

- Solutions to \( \dot{\mathbf{x}} = A\mathbf{x} \) are typically constructed from exponentials \( \mathbf{x}(t) = \frac{1}{2} \mathbf{v} e^{\mathbf{r}t} \)
  where \( \mathbf{r}, \mathbf{v} \) are eigenvalue/eigenvector pairs of \( A \). Here \( \mathbf{v} \) satisfies \( (A-\mathbf{r}I)\mathbf{v} = 0 \) for \( \mathbf{r} \) a solution to the characteristic equation of \( A \): \( \det(A-\mathbf{r}I) = 0 \).
General facts cont'd.

- If \( r = 0 \) is not a solution to \( \det(A - \lambda I) = 0 \), then \( \mathbf{x} = |L_0| \) is the only equilibrium solution of the ODE.

- Solutions are integral curves, parameterized by \( t \), in the \( x_1, x_2 \)-plane (phase plane) and a representative sample of curves is called a phase portrait.

- The general shape of the phase portrait determines the type of the equilibrium at \( L_0 \) and the direction of travel along solutions determines the stability of \( L_0 \).

- Both type and stability of \( L_0 \) are fully determined by the eigenvalues of \( A \) alone.
General facts cont'd.

- Classification of $x$ for $x' = Ax + x$: For $F = \mathbb{R}$, $F \neq 0$,

**Case I: $\gamma_1 \neq \gamma_2$ real**

Here, origin is a

- sink $\gamma_1 < 0, \gamma_2 < 0$ asymptotically stable
- source $\gamma_1 > 0, \gamma_2 > 0$ unstable
- saddle $\gamma_1 < 0, \gamma_2 > 0$, or $\gamma_1 > 0, \gamma_2 < 0$, also unstable

Notice that both a source and a saddle are unstable. But there is a difference. Can you characterize it?
Case II: $\gamma = \lambda + i\mu$, $\mu \neq 0$
Eigenvalues are complex conjugates

Here, origin is a
- spiral sink $\lambda < 0$, asymptotically stable
- spiral source $\lambda > 0$, unstable
- center $\lambda = 0$, stable
(no asymptotically stable)

Case III $\gamma_1 = \gamma_2$ (must be real)

Here, origin is a
- star node $\lambda > 0$, unstable
- $\lambda < 0$, asymptotically stable
- star sink

16 enough eigenvectors to construct 2 solutions (fixed)
- improper node 16 More or not enough eigenvectors to construct 2 lin. indep. solutions.
Many 2x2 systems of 1st order homogeneous ODEs are not linear (in the dependent variables)

\[ \begin{align*}
  \dot{x} &= F(x, y) \\
  \dot{y} &= G(x, y)
\end{align*} \]

although still autonomous (t is not explicit in the ODE, even as it always is in the solutions).

In vector form, (1) is written \( \mathbf{x}' = \mathbf{F}(\mathbf{x}) \),

where \( \mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \), \( \mathbf{F}(\mathbf{x}) = \begin{bmatrix} F(x, y) \\ G(x, y) \end{bmatrix} = \begin{bmatrix} F(x, y) \\ G(x, y) \end{bmatrix} \).

Note: Solutions to (1), written as \( \mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \)

are sometimes given a different notation like \( \mathbf{y}(t) = \begin{bmatrix} y(t) \\ z(t) \end{bmatrix} \) when the context encourages it.

Ex. We can write any \( \mathbf{x}' = A \mathbf{x} \) in the notation of (1), for \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \), as

\( F(x, y) = ax + by \)

\( G(x, y) = cx + dy \)
Note: In a non-linear system like (*), there can be more than one isolated singularity (equilibrium solution). Also the origin need not be an equilibrium.

Def. A pt $\overline{p} \in \mathbb{R}^2$ is a *critical point* of the system $\dot{x} = f(x)$ if $f(\overline{p}) = 0$. Critical pts are equilibrium of the system: $\dot{y}(t) = \overline{p}, \forall t \in \mathbb{R}$.

Note: The stability of a critical point $\overline{p}$ of $\dot{x} = f(x)$ is determined in the same way as that of a linear system; by determining how solutions behave around $\overline{p}$.

Recall the Lotka-Volterra eqns

\[
\begin{align*}
\dot{x} &= a_1 x - b_1 xy = f(x, y) \\
\dot{y} &= -a_2 y + b_2 xy = g(x, y)
\end{align*}
\]

$a_1, a_2, b_1, b_2 > 0$
ex. For $a_1 = 2, a_2 = 3, b_4 = 1, b_2 = 4$
Find all critical pts.

Solution: Here, ODE system is
\[ \begin{align*}
\dot{x} &= 2x - xy = x(2-y) = F(x,y) \\
\dot{y} &= -3y + 4xy = g(4x-3) = G(x,y)
\end{align*} \]

We look for $\dot{x} = 0 \iff F(x,y) = 0 = C(x,y)$.
Here \( C \) is domain is critical. Need to find another, assume $x \neq 0$. Then for $F(x,y) = 0$, we must have $y = 2$. Then if $y = 2$, for $G(x,y) = 0$, we must have $x = \frac{3}{4}$.

So $x = \frac{3}{4}, y = 2$ is also critical.

Thus are the only two.

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ex. Find all critical pts of $g$
\[ \begin{align*}
\dot{x} &= x - x^2 - xy \\
\dot{y} &= -y - y^2 + 2xy
\end{align*} \]
and discuss stability via the JODE phase portrait.

Note: This is Lotka-Volterra with extra terms.
Solution: We seek all solutions to system

\[ \begin{align*}
F(x, y) &= x - x^2 - xy = x(1-x-y) = 0 \\
G(x, y) &= -y - y^2 + 2xy = y(-1-y-2x) = 0.
\end{align*} \]

We do this by cases:

1. \( x = y = 0 \). The origin is critical.

2. Let \( y = 0 \) and assume \( x \neq 0 \). Then \( \frac{\partial G}{\partial x} = 0 \). For \( F(x, y) = 0 \) also, we need \( 1 - x - y = 0 = 1 - x \). Hence \( x = 1 \).

   Hence \( x = 1, y = 0 \) is critical.

3. Let \( x = 0 \) and assume \( y \neq 0 \). Now \( F(x, y) = 0 \) and to ensure \( \frac{\partial G}{\partial x} = 0 \) also, we need

   \[ -1 - y - 2x = -1 - y - 2(0) = 0. \]

   So \( y = -1 \).

   Hence \( x = 0, y = -1 \) is critical.
Assume $x \neq 0$ and $y \neq 0$. Then $F(x,y) = 0$ only if $1 - x - y = 0$ or along the line $y = 1 - x$. And $G(x,y) = 0$ only when $-1 - y - 2x = 0$, or along the line $y = 2x - 1$.

Both $F$ and $G$ are $0$ where these lines intersect: $x + y = 1$ solved by
$$
2x - y = 1 \quad \Rightarrow \quad x = \frac{2}{3}, \quad y = \frac{1}{3}.
$$

Hence $x = \frac{2}{3}, y = \frac{1}{3}$ is critical.

What do you see in the phase portrait?

- $[0,0]$, $[0,1]$ are saddles
- $[0,1]$ is an improper source?
- $[\frac{1}{3},\frac{2}{3}]$ is a spiral sink.

At this point, these are really just guesses.
2 more definitions

**Def.** Let the critical pt $\tilde{p}$ of $\tilde{x}' = \tilde{f}(x)$ be asymptotically stable (a sink). Then the set

$$B(\tilde{p}) = \left\{ x^0 \in \mathbb{R}^2 \mid \lim_{t \to \infty} \tilde{x}(t) = \tilde{p}, \tilde{x}(\epsilon t) = x^0 \right\}$$

is called the basin of attraction of $\tilde{p}$ under $\tilde{x}' = \tilde{f}(x)$. It is the set of all $x^0 \in \mathbb{R}^2$ whose solutions pass through an asymptote to $\tilde{p}$.

**Def.** If a solution $\tilde{x}(t)$ to $\tilde{x}' = \tilde{f}(x)$ comprises part of the boundary of a basin of attraction, it is called a separatrix.

Note: More strictly, hyperbolic or non-equilibrium noncritical solutions whose solution behavior is different on each side.
Lastly, a different viewpoint in phase plane analysis:

ex. Solve the system

\[ \begin{align*}
    \dot{x} &= 4 - 2y \\
    \dot{y} &= 12 - 3x^2
\end{align*} \]

Note: This system is non-linear (and non-homogeneous). Why is this so?

Solution: We can easily calculate the critical points here. \(4 - 2y = 0\) only when \(y = 2\). And \(12 - 3x^2 = 0\) when \(x = \pm 2\). Hence \(x = 2, y = 2\), and \(x = -2, y = 2\) are the only critical points.

But this is limited information.

Another tack? Remove the parameter \(t\) from the solutions by writing \(y\) directly (indirectly) in terms of \(x\). This involves a clever use of the Chain Rule (in Leibniz notation)
ex. For $a_1 = 2$, $a_2 = 3$, $b_4 = 1$, $b_2 = 4$

Find all critical pts.

Solution: Here, ODE system is

\[ \dot{x} = 2x - xy = x(2 - y) = f(x, y) \]
\[ \dot{y} = -3y + 4xy = y(4x - 3) = g(x, y) \]

We look for $\dot{\mathbf{x}} = [\dot{x}, \dot{y}]$ where $F(x, y) = 0 = G(x, y)$, here $\odot$ the origin is critical. Need to find another, assume $x \neq 0$. Then for $F(x, y) = 0$, we must have $y = 2$. Then if $y = 2$, for $G(x, y) = 0$, we must have $x = \frac{3}{4}$.

So $\odot$ $x = \frac{3}{4}$, $y = 2$ is also critical.

Here are the only two.

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ex. Find all critical pts of

\[ \dot{x} = x - x^2 - xy \]
\[ \dot{y} = -y - y^2 + 2xy \]

and discuss stability via the JODE

Note: This is the Lotka-Volterra model with extra terms. Called Competitor Species.

phase portrait.
Suppose we wanted to parametrize a curve given by \( y = g(x) \). Write \( x \) as some function of \( t \). Then \( y \) is also a function of \( t \) given by \( y(t) = g(x(t)) \). By diff., we get \( y'(t) = g'(x(t)) \cdot x'(t) \), or in Leibniz notation \( \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \), so \( \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \).

We can rewrite this process and write this system of ODEs into a single 1st order ODE:

1. \( x' = \frac{dx}{dt} = 4 - 2y \)
2. \( y' = \frac{dy}{dt} = 12 - 3x^2 \)

One can solve this as an exact ODE given by

\[ \frac{dy}{dx} - \frac{12 - 3x^2}{4 - 2y} = 0 , \quad or \quad (4 - 2y) \frac{dy}{dx} - (12 - 3x^2) = 0 \]

solved to give \( y(x, y) = 4y - y^2 - 12x + x^3 = C \).
Level sets of $y(x,v)$ correspond to solution curves:

more general version of a separatrix