110.302 Lecture 28 :

In preparation for the study of non-linear systems, Section 9.1 is a review of the linear systems theory, but given solely in terms of understanding how to characterize the type and stability of the equilibrium solution at the origin of the phase plane.

<u>General facts</u>

- Solutions to $\vec{x}\,' = A_{2\times 2}\vec{x}$ are typically constructed from exponentials $\vec{x}(t) = \vec{v}e^{rt}$ where $r, \vec{v}$ are eigenvalue/eigenvector pairs of $A$. Here $\vec{v}$ satisfies $(A - rI)\vec{v} = \vec{0}$ for $r$ a solution to the characteristic equation of $A$: $\det(A - rI) = 0$.

General facts cont'd.

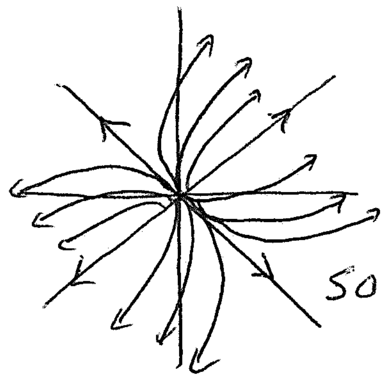
- If $r=0$ is not a solution to $\det(A-rI)=0$, then $\vec{x}=\begin{bmatrix}0\\0\end{bmatrix}$ is the only equilibrium solution of the ODE
- Solutions are integral curves, parameterized by $t$, in the $x_1x_2$-plane (phase plane) and a representative sample of curves is called a phase portrait.
- The general shape of the phase portrait determines the <u>type</u> of the equilibrium at $\begin{bmatrix}0\\0\end{bmatrix}$ and the direction of travel along solutions determines the stability of $\begin{bmatrix}0\\0\end{bmatrix}$.
- Both type and stability of $\begin{bmatrix}0\\0\end{bmatrix}$ are fully determined by the eigenvalues of $A$ alone.

## General facts cont'd.

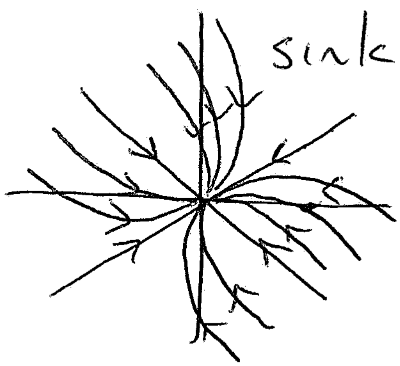
• Classification of $\begin{bmatrix}0\\0\end{bmatrix}$ for $\vec{X}' = A_{2\times 2}\vec{x}$ : for $r \neq 0$,

**Case I** : $r_1 \neq r_2$ real

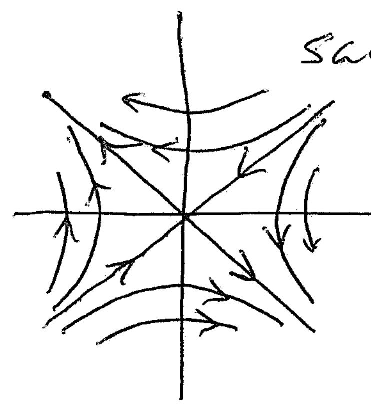
Here, origin is a
- • sink $r_1 < 0,\ r_2 < 0$ asympt. stable
- • source $r_1 > 0,\ r_2 > 0$ unstable
- • saddle $r_1 < 0,\ r_2 > 0$, or $r_1 > 0,\ r_2 < 0$. also unstable



source



sink



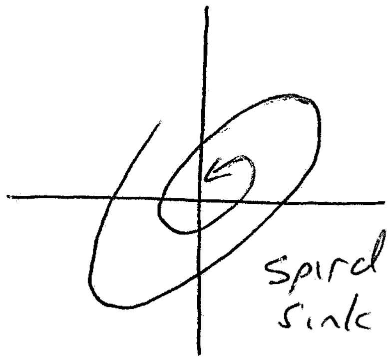
saddle

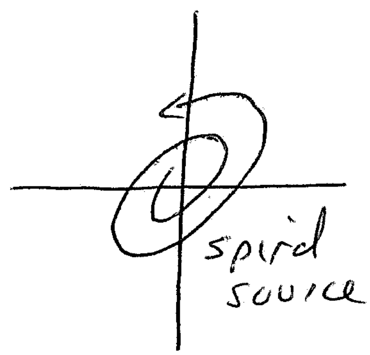
Notice that both a source and a saddle are unstable. But there is a difference. Can you characterize it?

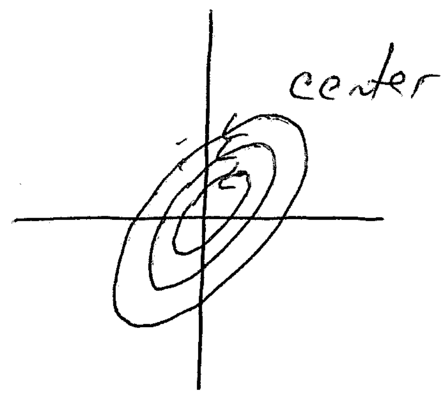
**Case II:** $r = \lambda \pm i\mu$, $\mu \neq 0$

Eigenvalues are complex conjugates

Here, origin is a
- spiral sink $\lambda < 0$ asympt. stable
- spiral source $\lambda > 0$ unstable
- center $\lambda = 0$ stable (not asymptotically stable)

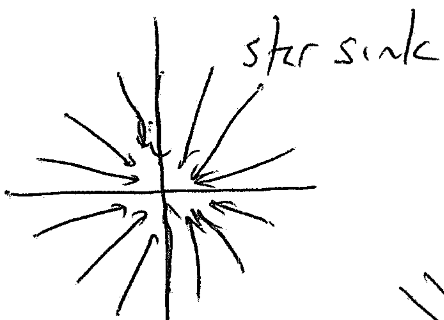


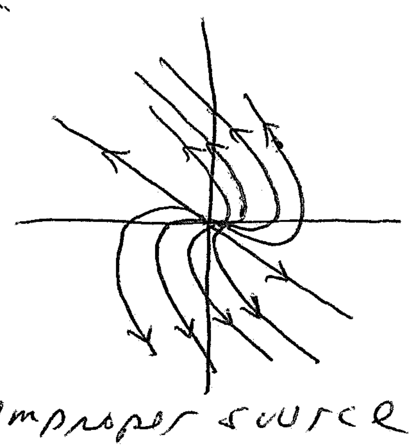




**Case III** $r_1 = r_2$ (must be real)

Here, origin is a
- star node $r > 0$ unstable, $r < 0$ asympt stable, if enough eigenvectors to construct 2 solutions (lin ind)
- improper node if there are not enough eigenvectors to construct 2 lin. indep. solutions.



Many 2x2 systems of $1^{st}$ order homogenous ODEs ~~a~~ are not linear (in the dependent variables)

$$(*) \qquad \begin{aligned} \dot{x} &= F(x,y) \\ \dot{y} &= G(x,y) \end{aligned}$$

although still autonomous ($t$ is not explicit in the ODE, even as it always in in the solutions).

In vector form, $(*)$ is written $\vec{x}' = \vec{f}(\vec{x})$,

where $\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$, $\vec{f}(\vec{x}) = \begin{bmatrix} F(\vec{x}) \\ G(\vec{x}) \end{bmatrix} = \begin{bmatrix} F(x,y) \\ G(x,y) \end{bmatrix}$.

Note: Solutions to $(*)$, written as $\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ are sometimes given a different notation like $\vec{\varphi}(t) = \begin{bmatrix} \varphi(t) \\ \psi(t) \end{bmatrix}$ when the context encourages it.

ex. We can write any $\vec{x}' = A\vec{x}$ in the notation of $(*)$, for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, as

$$\begin{aligned} F(x,y) &= ax + by \\ G(x,y) &= cx + dy \end{aligned}$$

Note: In a non-linear system like (*), there can be more than one isolated singularity (equilibrium solution). Also the origin need not be an equilibrium.

<u>Def</u>. A pt $\vec{p} \in \mathbb{R}^2$ is a <u>critical pt</u> of the system $\vec{x}' = \vec{f}(\vec{x})$ if $\vec{f}(\vec{p}) = \vec{0}$. Critical pts are equilibria of the system: $\vec{\varphi}(t) = \vec{p}$. $\forall t \in \mathbb{R}$.

<u>Note</u>: The stability of a critical pt $\vec{p}$ of $\vec{x}' = \vec{f}(\vec{x})$ is determined in the same way as that of a linear system; by determining how solutions behave <u>around</u> $\vec{p}$.

Recall the Lotka-Volterra eqns

$$\dot{x} = a_1 x - b_1 xy = F(x,y)$$
$$\dot{y} = -a_2 y + b_2 xy = G(x,y)$$

$a_1, a_2, b_1, b_2 > 0$

ex. For $a_1 = 2,\ a_2 = 3,\ b_1 = 1,\ b_2 = 4$
Find all critical pts.

<u>Solution</u>: Here, ODE system is

$$\dot{x} = 2x - xy = x(2-y) = F(x,y)$$
$$\dot{y} = -3y + 4xy = y(4x-3) = G(x,y)$$

We look for $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ where $F(x,y) = 0 = G(x,y)$.

Here ① The origin is critical. ~~And~~ To find another, assume $x \neq 0$. Then for $F(x,y) = 0$, we must have $y = 2$. Then if $y = 2$, for $G(x,y) = 0$, we must have $x = \frac{3}{4}$.

So ② $x = \frac{3}{4},\ y = 2$ is also critical.

These are the only two. ∎

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ex. Find all critical pts of

$$\dot{x} = x - x^2 - xy$$
$$\dot{y} = -y - y^2 + 2xy$$

Note: This is Lotka-Volterra with extra terms.

and discuss stability via the JODE phase ~~de~~ portrait.

Solution: We seek all solutions to system

$$F(x,y) = x - x^2 - xy = x(1-x-y) = 0$$
$$G(x,y) = -y - y^2 + 2xy = y(-1-y-2x) = 0.$$

We do this by cases:

① $x = y = 0$. The origin is critical.

② Let $y=0$ and assume $x \neq 0$. Then $G(x,y) = 0$. For $F(x,y) = 0$ also, we need $1 - x - y = 0 = 1 - x$. Here $x = 1$.

Hence $x = 1$, $y = 0$ is critical.

③ Let $x = 0$ and assume $y \neq 0$. Then $F(x,y) = 0$ and to ensure $G(x,y) = 0$ also, we need $-1 - y - 2x = -1 - y - 2(0) = 0$. So $y = -1$.

Hence $x = 0$, $y = -1$ is critical.

Soln cont'd.

(4) Assume $x \neq 0$ and $y \neq 0$. Then $F(x,y)=0$ only if $1-x-y=0$ or along the line $y=1-x$. And $G(x,y)=0$ only when $-1-y-2x=0$, or along the line $y=2x-1$.

Both F and G are 0 where these lines intersect:
$$\left.\begin{array}{r} x+y=1 \\ 2x-y=1 \end{array}\right\} \text{ solved by } x=\tfrac{2}{3} \;\; y=\tfrac{1}{3}.$$

Hence $x=\frac{2}{3}$, $y=\frac{1}{3}$ is critical.

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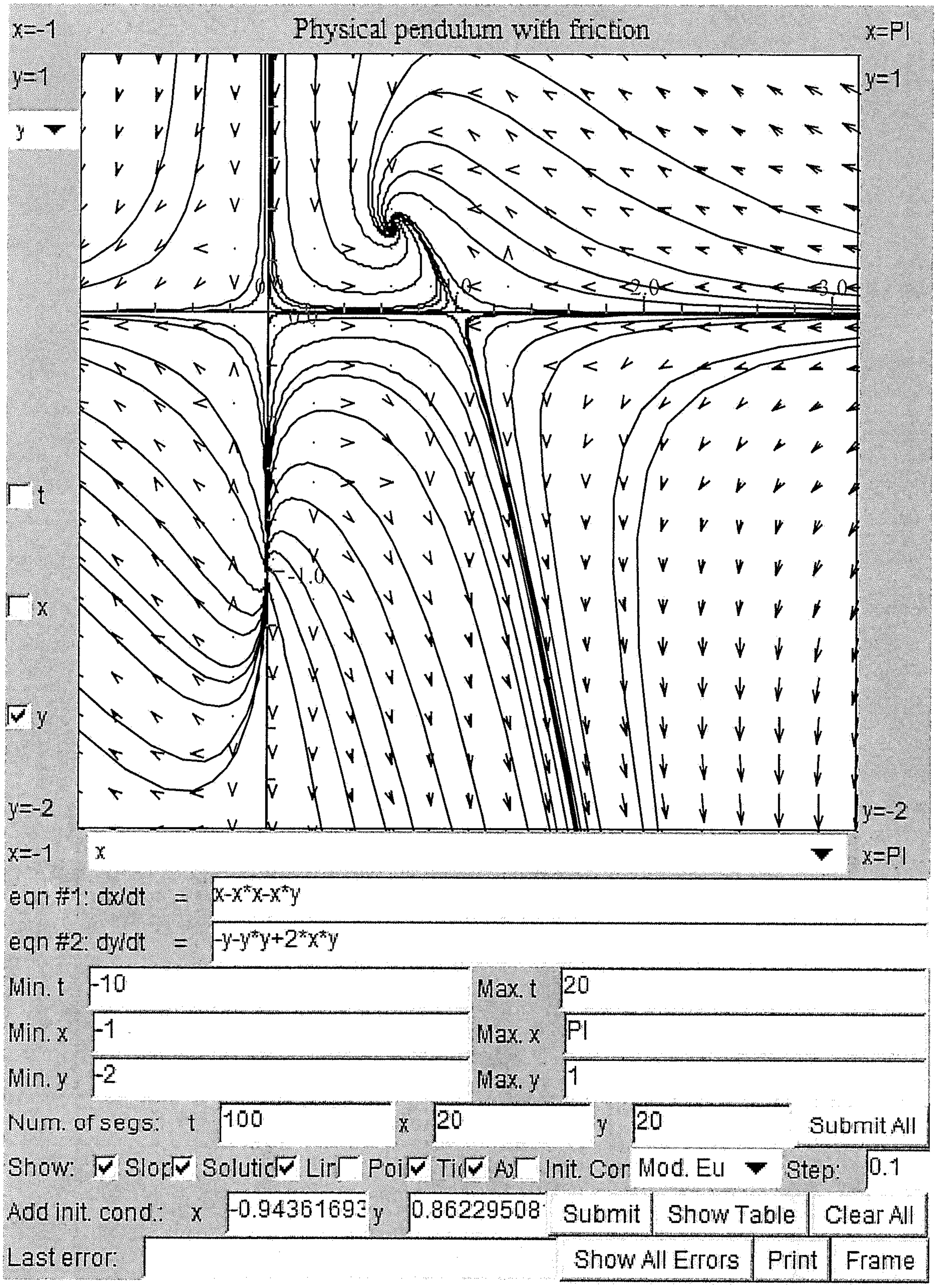
What do you see in the phase portrait?

- $\begin{bmatrix}0\\0\end{bmatrix}$, $\begin{bmatrix}1\\0\end{bmatrix}$ are saddles
- $\begin{bmatrix}0\\-1\end{bmatrix}$ is ~~unspecified sink~~ an improper ~~sink~~ source?
- $\begin{bmatrix}2/3\\1/3\end{bmatrix}$ is ~~an improper~~ a spiral sink.

At this point, these are really just guesses.

X



Physical pendulum with friction
x=-1
x=PI
y=1
y=-2
t
x
y
eqn #1: dx/dt = x-x*x-x*y
eqn #2: dy/dt = -y-y*y+2*x*y
Min. t -10
Max. t 20
Min. x -1
Max. x PI
Min. y -2
Max. y 1
Num. of segs: t 100 x 20 y 20
Submit All
Show: Slop Solutic Lin Poi Ti A Init. Con
Mod. Eu
Step: 0.1
Add init. cond.: x -0.94361693 y 0.86229508
Submit
Show Table
Clear All
Last error:
Show All Errors
Print
Frame

2 more definitions

**Def.** Let the critical pt $\vec{p}$ of $\vec{x}' = \vec{f(\vec{x})}$ be asymptotically stable (a sink). Then the set

$$B(\vec{p}) = \left\{ \vec{x}^\circ \in \mathbb{R}^2 \;\middle|\; \lim_{t\to\infty} \vec{x}(t) = \vec{p} \quad \vec{x}(t_0) = \vec{x}^\circ \right\}.$$

is called the basin of attraction of $\vec{p}$ under $\vec{x}' = \vec{f(\vec{x})}$. It is the set of pts in ~~$\mathbb{R}^2$~~ the phaseplane whose solutions passing through are asymptotic to $\vec{p}$.

**Def** If a solution $\vec{x}(t)$ to $\vec{x}' = \vec{f(\vec{x})}$ comprises part of the ~~edge~~ boundary of a basin of attraction, it is called a separatrix.

Note: More broadly, separatrices are ~~non equilibrium~~ noncritical solutions where solution behavior is different on each side.

110.302 Lecture 29: XII

Lastly, a different viewpoint in phase plane analysis:

ex. Solve the system $\dot{x} = 4-2y$
$$\dot{y} = 12-3x^2$$

Note: This system is nonlinear (and nonhomogeneous ???) (why?). It is still autonomous. What does this mean?

Solution: We can easily calculate the critical pts here. $4-2y=0$ only when $y=2$. And $12-3x^2=0$ when $x=\pm 2$. Hence $x=2, y=2$, and $x=-2, y=2$ are the only critical pts.

But this is limited information.

Another tack? Remove the parameter $t$ from the solutions by writing $y$ directly (implicitly) in terms of $x$: This involves a creative use of the Chain Rule (in Leibniz notation)

ex. For $a_1 = 2$, $a_2 = 3$, $b_1 = 1$, $b_2 = 4$
Find all critical pts.

Solution: Here, ODE system is

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ex. Find all critical pts of

$$\dot{x} = x - x^2 - xy$$
$$\dot{y} = -y - y^2 + 2xy$$

Note: This is Lotka-Volterra with extra terms. Called Competing Species.

and discuss stability via the JODE phase ~~de~~ portrait.

Suppose we wanted to parameterize a curve given by $y = y(x)$. Write $x$ as some function of $t$. Then $y$ is also a function of $t$ given by $y(t) = y(x(t))$. If diff, then $y'(t) = y'(x(t)) \cdot x'(t)$, or in Leibniz notation $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$, so $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

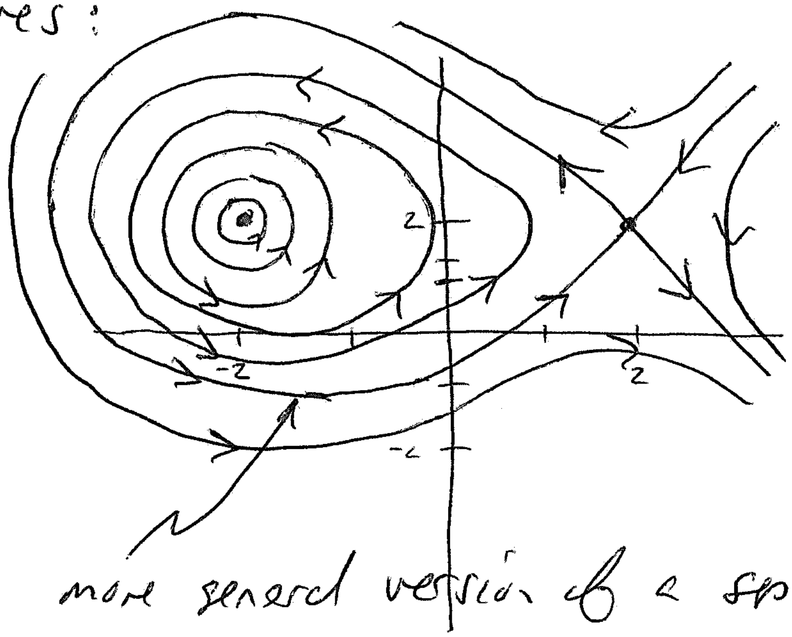
We can reverse this process and write this system of ODE into a single 1st order ODE: $\dot{x} = \frac{dx}{dt} = 4 - 2y$ $\dot{y} = \frac{dy}{dt} = 12 - 3x^2$ into

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{12 - 3x^2}{4 - 2y}$$

One may notice this as an exact ODE given by $\frac{dy}{dx} - \frac{12-3x^2}{4-2y} = 0$, or $(4-2y)\frac{dy}{dx} - (12 - 3x^2) = 0$ solved to give $\Psi(x,y) = 4y - y^2 - 12x + x^3 = c$.

XIV

Level sets of $\psi(x,y)$ correspond to solution curves:




more general version of a separatrix